CHAPTER 2

THE MOMENTUM PRINCIPLE: IMPULSE AND MOMENTUM CHANGE

Key concepts

- One or more objects can be considered to be a "system."
  - Everything not in the system is part of the "surroundings."
- The momentum of a system can be changed only by interactions with the surroundings.
  - Force is a quantitative measure of interaction.
  - Force is a vector (it has magnitude and direction).
- The change in momentum of a system is equal to the net impulse applied, which involves:
  - The total force exerted on it by all objects not in the system
  - The time interval over which the interaction occurs.
- Momentum is *conserved*: The change in momentum of a system plus the change in momentum of its surroundings is zero.

The Momentum Principle is the first of three fundamental principles of mechanics which together make it possible to predict and explain a very broad range of real-world phenomena (the other two are the Energy Principle and the Angular Momentum Principle). The Momentum Principle makes a quantitative connection between amount of interaction and change of momentum.

A fundamental principle relates the change in a quantity to the interaction causing the change, and is special because it applies in absolutely every situation.

2.1 SYSTEM AND SURROUNDINGS

One or more objects can be considered to be a "system." Everything that is not included in the system is part of the "surroundings." The Momentum Principle relates the change in momentum of a system to the amount of interaction with its surroundings. No matter what system we choose, the Momentum Principle will correctly predict the behavior of the system.

As an example, consider the following situation: We are in a spacecraft in outer space, far from massive objects such as planets. In the spacecraft are two identical small, hard balls, each with a mass of 2 kg. Initially ball B is at rest, and ball A is moving toward ball B with velocity $\langle 3,0,0\rangle$ m/s, as shown in Figure 2.1, so its momentum is

$$\vec{p}_A = (2\,\text{kg})\langle 3,0,0\rangle\,\text{m/s} = \langle 6,0,0\rangle\,\text{kg}\cdot\text{m/s}$$

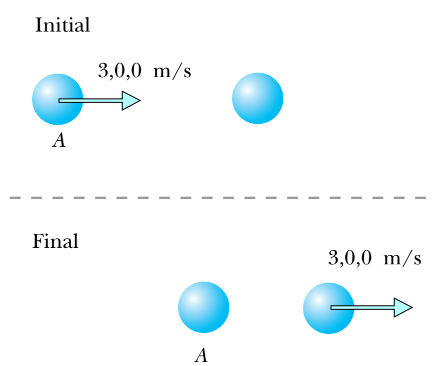


Figure 2.1 A head-on collision between two identical balls. Initially B is at rest. After the collision A is at rest.

If A collides exactly head-on with B, A stops dead, and B is now observed to move with velocity $\langle 3,0,0\rangle$ m/s, and momentum $\langle 6,0,0\rangle$ kg · m/s.

System: One object (ball A)

In thinking about the collision described above, we could choose ball A to be the system, and ball B to be part of the surroundings. In this case, we would say that the momentum of ball A changed because ball A interacted with its surroundings (ball B). The change of momentum of ball A was:

$$\Delta\vec{p}_A = \langle 0,0,0\rangle\,\text{kg}\cdot\text{m/s} - \langle 6,0,0\rangle\,\text{kg}\cdot\text{m/s} = \langle -6,0,0\rangle\,\text{kg}\cdot\text{m/s}$$

The momentum of the surroundings (ball B) also changed:

$$\Delta\vec{p}_B = \langle 6,0,0\rangle\,\text{kg}\cdot\text{m/s} - \langle 0,0,0\rangle\,\text{kg}\cdot\text{m/s} = \langle +6,0,0\rangle\,\text{kg}\cdot\text{m/s}$$

In a sense, momentum is "sticky"—it's hard to get rid of. Once a system has momentum, the only way to change that momentum is through an interaction which transfers momentum to the surroundings.

The sum of the change of momentum of the system (A) and the change of momentum of the surroundings (B) is zero.

System: Both objects (ball A and ball B)

Alternatively, we could choose to include both A and B in the system; in this case the surroundings include the air in the spacecraft, the spacecraft itself, etc. In this case, at every instant we need to consider the total momentum of the system, which is simply the sum of the momenta of balls A and B.

Initially, the momentum of the two-ball system was:

$$\vec{p}_i = \langle 6,0,0 \rangle \text{ kg}\cdot\text{m/s} + \langle 0,0,0 \rangle \text{ kg}\cdot\text{m/s} = \langle 6,0,0 \rangle \text{ kg}\cdot\text{m/s}$$

After the collision, the momentum of the two-ball system was:

$$\vec{p}_f = \langle 0,0,0 \rangle \text{ kg}\cdot\text{m/s} + \langle 6,0,0 \rangle \text{ kg}\cdot\text{m/s} = \langle 6,0,0 \rangle \text{ kg}\cdot\text{m/s}$$

The change in momentum of the two-ball system was $\vec{0}$, because the system did not interact significantly with its surroundings (momentum simply "flowed" from one object within the system to another). The change in momentum of the surroundings was also $\vec{0}$, so the sum of these changes was $\vec{0}$.

Because the sum of the change of momentum of the system and the change of momentum of the surroundings is always zero, we say that momentum is a "conserved" quantity. We will discuss this in more detail in Section 2.10.

2.2 THE MOMENTUM PRINCIPLE

The Momentum Principle is a fundamental principle that is sometimes called Newton's second law. It restates and extends Newton's first law of motion in a quantitative, causal form that can be used to predict the behavior of objects. The validity of the Momentum Principle has been verified through a very wide variety of observations and experiments, involving large and small objects, moving slowly or at speeds near the speed of light. It is a summary of the way interactions affect motion in the real world.

The Momentum Principle is a Fundamental Principle because:

It applies to every possible system, no matter how large or small (from clusters of galaxies to subatomic particles), or how fast it is moving.

It is true for any kind of interaction (electric, gravitational, etc.)

It relates an effect (change in momentum) to a cause (an interaction).

THE MOMENTUM PRINCIPLE

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t$$

In words: change of momentum of a system (the effect) is equal to the net force acting on the system times the duration of the interaction (the cause).

As usual, the capital Greek letter delta (Δ) means "change of" (something), or "final minus initial." We will see that Δt must be small enough that the net force on the system does not change significantly during the interval.

$\Delta\vec{p}$ $\quad \vec{F}_{\text{net}}\,\Delta t$

Change of momentum

As we saw in the previous chapter, the change in momentum of a system can involve

- a change in the magnitude of momentum
- a change in the direction of momentum
- change in both magnitude and direction

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

You are already familiar with change of momentum $\Delta\vec{p}$ and with time interval Δt. The new element is the concept of "force." The "net" force $\vec{F}_{\text{net}}$ is the vector sum of all the forces acting on an object.

$\Delta\vec{p} \quad \vec{F}_{\text{net}}\,\Delta t$

Net force

Scientists and engineers employ the concept of "force" to quantify interactions between two objects. Force is a vector quantity because a force has a magnitude and is exerted in a particular direction. Examples of forces include the following:

- the repulsive electric force a proton exerts on another proton
- the attractive gravitational force the Earth exerts on you
- the force that a compressed spring exerts on your hand
- the force on a spacecraft of expanding gases in a rocket engine
- the force of the air on the propeller of an airplane or swamp boat

Measuring the magnitude of the velocity of an object (in other words, measuring its speed) is a familiar task, but how do we measure the magnitude of a force?

A simple way to measure force is to use the stretch or compression of a spring. In Figure 2.2 we hang a block from a spring, and note that the spring is stretched a distance s. Then we hang two such blocks from the spring, and we see that the spring is stretched twice as much. By experimentation, we find that any spring made of the same material and produced to the same specifications behaves in the same way. Similarly, we can observe how much the spring compresses when the same blocks are supported by it (Figure 2.3). We find that one block compresses the spring by the same distance $|s|$, and two blocks compress it by $2|s|$. (Compression can be considered negative stretch, because the length of the spring decreases.)

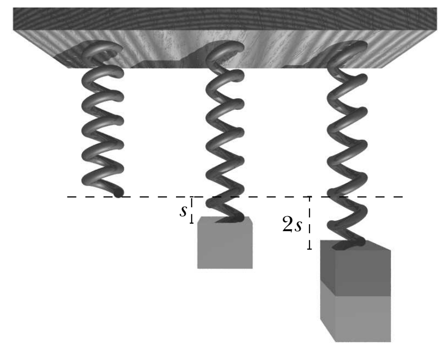


Figure 2.2 Stretching of a spring is a measure of force.

We can use a spring to make a scale for measuring forces, calibrating it in terms of what force is required to produce a given stretch. The SI unit of force is the "newton," abbreviated as "N." One newton is a rather small force. A newton is approximately the downward gravitational force of the Earth on a small apple, or about a quarter of a pound. If you hold a small apple at rest in your hand, you apply an upward force of about one newton, compensating for the downward pull of the Earth.

The net force, the vector sum of all the forces acting on a system, acting for some time Δt causes changes of momentum.

DEFINITION OF NET FORCE

$$\vec{F}_{\text{net}} \equiv \vec{F}_1 + \vec{F}_2 + \cdots$$

"Net" means "total". The net force acting on a system is the vector sum of all forces acting on the system.

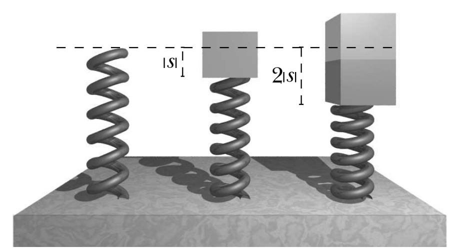


Figure 2.3 Compression of a spring is also a measure of force.

The net force acting on a system is the vector sum of all of the forces exerted on the system by objects in the surroundings. Forces internal to the system may exist, but cannot change the momentum of the system. See the discussion and proof in Section 2.8.

We find experimentally that the magnitude of the net force acting on an object affects the magnitude of the change in its momentum. Many introductory physics laboratories have air tracks like the one illustrated in Figure 2.4. The long triangular base has many small holes in it, and air under pressure is blown out through these holes. The air forms a cushion under the glider, allowing it to coast smoothly with very little friction.

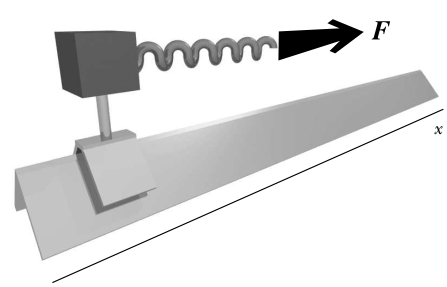


Figure 2.4 A block mounted on a nearly frictionless air track, pulled by a spring.

Suppose we place a block on a glider on a long air track, and attach a calibrated spring to it (Figure 2.4). We pull on the spring so that it is stretched a distance s, so that it exerts a force F on the block, and we pull for a short time. We observe that the momentum of the block increases from zero to an amount mv (since $v \ll c$). If instead we pull on the spring so that it stretches a distance $2s$, the spring exerts a force $2F$ on the block, and we observe that the block's momentum increases to $2mv$ in the same amount of time. Apparently the magnitude of the change in momentum is proportional to the magnitude of the net force applied to the object.

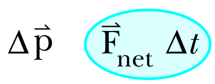
Duration of interaction

Another experiment will show us that not only the magnitude of the force, but the length of time during which it acts on an object, affects the change

$\Delta\vec{p}$ $\vec{F}_{\text{net}}\Delta t$

of momentum of the object. Suppose we again place a block on an air track and attach a spring to it. We pull on the spring for a time Δt with a force F, and we observe that the momentum of the block increases from zero to an amount mv. However, if we repeat the experiment but pull for a time twice as long with the same force F, the block's momentum increases to $2mv$; the change in its momentum is twice as great. We observe that the magnitude of the change in momentum is directly proportional to the length of time $\Delta t = t_2 - t_1$ during which the force acts on the object.

Impulse



The amount of interaction affecting an object includes both the strength of the interaction (expressed as the net force $\vec{F}_{net}$) and the duration Δt of the interaction. Either a bigger force, or applying the force for a longer time, will cause more change of momentum.

The product of a force and a time interval is called "impulse".

DEFINITION OF IMPULSE

$$\text{Impulse} \equiv \vec{F}\Delta t \text{ (for small enough } \Delta t)$$

Impulse has units of N · s (newton-seconds)

With this definition of impulse we can state the Momentum Principle in words like this:

The change of momentum of a system is equal to the net impulse applied to it.

What if the force is not constant during the interval Δt?

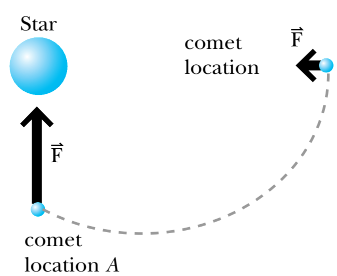


Figure 2.5 The direction and magnitude of the gravitational force on a comet by a star change as the comet's position changes. The size of the comet is exaggerated in this diagram.

In the air track experiment described above, we were able to keep the force constant during the time interval Δt, by keeping the stretch of the spring constant. In many cases (perhaps most real-world cases), the force applied to an object is not constant, but changes as the object moves. For example, the magnitude of the force exerted on an object by a spring attached to the object changes when the stretch of the spring changes. As a comet moves, both the direction and the magnitude of the force exerted on the comet by a star changes (Figure 2.5).

If the magnitude or the direction of a force changes during a time interval Δt, what value of the force should we use in calculating the impulse? There are two possibilities:

- Use an approximate average value of the force
- Divide the time interval into several time intervals small enough that the force remains approximately constant over each smaller time interval

Each of these possibilities requires that we make an approximation. This is fine, but it is important to be aware that we are making an approximation!

Δt must be small enough that the force can be considered to be approximately constant over the time interval.

This is the same issue we met with the position update relation, $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t$, where we need to use a short enough time interval that the velocity isn't changing very much, or else we need to know the average velocity during the time interval.

? A constant net force $\langle 3, -5, 4\rangle$ N acts on an object for 10 s. What is the net impulse applied to the object? What was the change in momentum of the object?

$$\text{impulse} = \vec{F}_{net}\Delta t = \langle 3, -5, 4\rangle\,\text{N}\cdot(10\,\text{s}) = \langle 30, -50, 40\rangle\,\text{N}\cdot\text{s}$$

A newton can be expressed in terms of kilograms, meters, and seconds.
$1\,\text{N} = 1\,\text{kg}\cdot\text{m/s}^2$

The change in the object's momentum is equal to the net impulse, so

$$\Delta\vec{p} = \langle 30, -50, 40\rangle\,\text{kg}\cdot\text{m/s}$$

Evidently $1\,\text{N}\cdot\text{s} = 1\,\text{kg}\cdot\text{m/s}$, so $1\,\text{N} = 1\,\text{kg}\cdot\text{m/s}^2$.

2.X.1 A constant net force of $\langle -0.5, -0.2, 0.8\rangle$ N acts on an object for 2 minutes. (a) What is the impulse applied to the object, in SI units? (b) What is the change in the momentum of the object?

2.X.2 A hockey puck initially has momentum $\langle 0, 2, 0\rangle$ kg · m/s. It slides along the ice, gradually slowing down, until it comes to a stop. (a) What was the impulse applied by the ice to the hockey puck? (b) It took 3 seconds for the puck to come to a stop. During this time interval, what was the net force on the puck by the ice and the air (assuming this force was constant)?

Update form of the Momentum Principle

Since $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{F}_{\text{net}}\Delta t$ ("final minus initial"), we can rearrange the Momentum Principle to the update form:

THE MOMENTUM PRINCIPLE (UPDATE FORM)

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

for a time interval Δt short enough that the net force is approximately constant over this time interval

This update version of the Momentum Principle emphasizes the fact that if you know the initial momentum, and you know the net force acting during a "short enough" time interval, you can predict the final momentum.

The "update form" of the Momentum Principle can be used to predict the new momentum of a system to which a known force has been applied for a known time interval.

Separation of components

A great deal of information is expressed compactly in the vector form of this equation, $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$. The Momentum Principle written in terms of vectors can be interpreted as three ordinary scalar equations, for components of the motion along the x, y, and z axes:

$$p_{fx} = p_{ix} + F_{\text{net},x}\Delta t$$
$$p_{fy} = p_{iy} + F_{\text{net},y}\Delta t$$
$$p_{fz} = p_{iz} + F_{\text{net},z}\Delta t$$

An important aspect of this is the fact that the x component of an object's momentum cannot be affected by forces in the y or z directions.

This fact can be very useful in solving problems. In some situations, for example, if we know that the y and z components of an object's momentum are not changing, we may choose to work only with the x component of the momentum update equation.

Applying the Momentum Principle

We can use the Momentum Principle to predict the change in momentum of an object, in an experiment using the air track apparatus described above. We will choose the x axis to point in the direction of the motion. We stretch a calibrated spring, which exerts a force on the block of 20 N when stretched 4 cm. Suppose the block starts from rest, and the spring pulls the block for 1 second in the $+x$ direction with a force of magnitude 20 N (you will have to move forward in order to keep the spring stretched).

? The block starts from rest, so $\vec{p}_i = \langle 0, 0, 0\rangle$ kg · m/s. What would the Momentum Principle predict the new momentum of the block to be after 1 second?

Since the friction force on the glider is negligibly small, the net force on the glider is just the force exerted by the spring. The Momentum Principle (update form) applied to the glider is:

$$\vec{p}_f = \langle 0, 0, 0\rangle \text{ kg}\cdot\text{m/s} + \langle 20, 0, 0\rangle \text{ N}(1\text{ s})$$

Since the net force is in the x direction, we know that the y and z components of the glider's momentum will not change, and we can work with just the x component of the momentum.

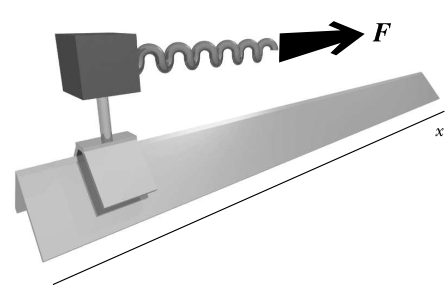


Figure 2.6 Apply a constant force to a block on a low-friction air track.

$$p_{fx} = p_{ix} + F_{\text{net},x}\Delta t = 0 + (20\,\text{N})(1\,\text{s}) = 20\,\text{kg}\cdot\text{m/s}$$

If you do the experiment, this is what you will observe.

Suppose the spring keeps exerting a force on the block for another second, but now with the spring stretched half as much, 2 cm, so you know that the spring is exerting half the original force (10 N) on the block.

> ? What would the Momentum Principle predict the new x component of the momentum of the block to be now?

The Momentum Principle would predict the following, where we take the final momentum from the first pull and consider that to be the initial momentum for the second pull:

$$p_{fx} = p_{ix} + F_{\text{net},x}\Delta t = (20\,\text{kg}\cdot\text{m/s}) + (10\,\text{N})(1\,\text{s}) = 30\,\text{kg}\cdot\text{m/s}$$

If you do the experiment, this is what you will observe (Figure 2.7). Note that the effects of the interactions in the two 1-second intervals add; we add the two momentum changes.

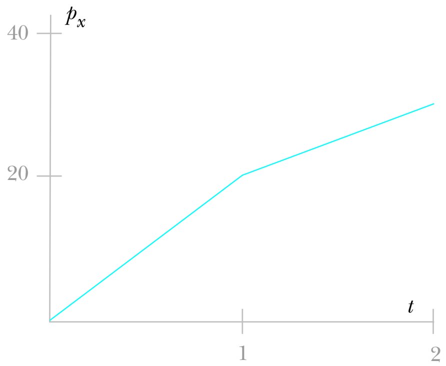


Figure 2.7 x component of momentum of the block (kg · m/s) vs. time (s), when pulled for 1 second with a force of 20 N, then for 1 second with a force of 10 N.

Instead of varying the net force, we could try varying the duration of the interaction. Start over with the block initially at rest. Pull for 2 seconds with the spring stretched 4 cm, so the force of the spring on the block is 20 N.

> ? The block starts from rest ($p_{ix} = 0$). What would the Momentum Principle predict the new x component of the momentum of the block to be?

$$p_{fx} = p_{ix} + F_{\text{net},x}\Delta t = 0 + (20\,\text{N})(2\,\text{s}) = 40\,\text{kg}\cdot\text{m/s}$$

Here (Figure 2.8) the final x component of momentum was 40 kg · m/s after applying a force of 20 N for 2 s, whereas in the previous experiments we got 30 kg · m/s after the spring applied a force of 20 N for 1 s plus a force of 10 N for 1 s. In our calculations we can use big values of Δt as long as the force isn't changing much. When the force changed from 20 N for a second to 10 N for a second we had to treat the two time intervals separately.

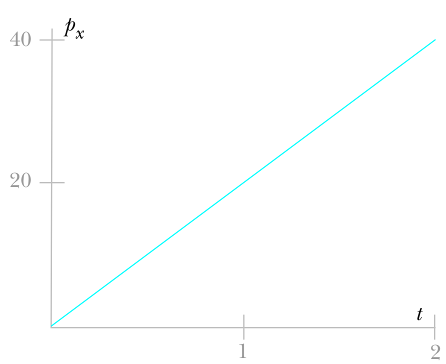


Figure 2.8 x component of momentum of the block (kg · m/s) vs. time (s), when pulled for 2 seconds with a force of 20 N.

Many different experiments have shown the validity of the Momentum Principle. If we use two springs to move the block, we find that it is indeed the vector sum of the two spring forces, the "net" force, that accounts for the change in momentum.

2.3 APPLYING THE MOMENTUM PRINCIPLE

To apply the Momentum Principle to a real-world situation, several steps are required:

- Choose a system, consisting of a portion of the Universe. The rest of the Universe is called the surroundings.
- Make a list of the objects in the surroundings that exert significant forces on the chosen system, and make a labeled diagram showing the external forces exerted by the objects in the surroundings.
- Choose initial and final times (often these are obvious; in examples we will comment on this when necessary).
- Apply the Momentum Principle to the chosen system:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

  Substitute known values into the terms of the Momentum Principle.
- Check units; check the reasonableness of your answer (direction of momentum, change in magnitude of momentum), based on the physical situation.

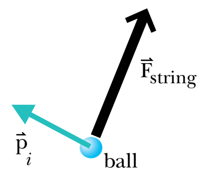
Example: Ball and string (2D, single force)

Inside a spaceship in outer space there is a small steel ball of mass 0.25 kg. At a particular instant, the ball has momentum $\langle -8, 3, 0 \rangle$ kg·m/s. At this instant the ball is being pulled by a string, which exerts a net force $\langle 10, 25, 0 \rangle$ N on the ball. What is the ball's approximate momentum 0.5 seconds later?

Assumption: The force didn't change much during 0.5 second.

System: the steel ball
Surroundings: the string

Momentum Principle:



$$\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$$
$$\vec{p}_f = (\langle -8, 3, 0 \rangle \text{ kg} \cdot \text{m/s}) + (\langle 10, 25, 0 \rangle \text{ N})(0.5 \text{ s})$$
$$\vec{p}_f = (\langle -8, 3, 0 \rangle \text{ kg} \cdot \text{m/s}) + (\langle 5, 12.5, 0 \rangle \text{ N} \cdot \text{s})$$
$$\vec{p}_f = \langle -3, 15.5, 0 \rangle \text{ kg} \cdot \text{m/s}$$

Check: units correct (momentum: kg · m/s and position: m); direction of final momentum makes sense (Figure 2.9).

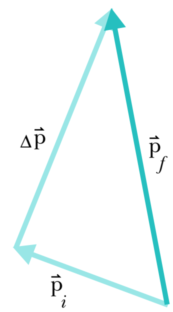


Figure 2.9 Initial momentum, change in momentum, and final momentum for a ball pulled by a string.

More than one force

An easy way to arrange to apply a nearly constant force is to mount a battery powered fan on a cart (Figure 2.10). If the fan directs air backwards, the interaction with the air pushes the cart forward with a nearly constant force. Swamp boats used in the very shallow Florida Everglades are built in a similar way, with large fans on top of the boats propelling them through the swamp.

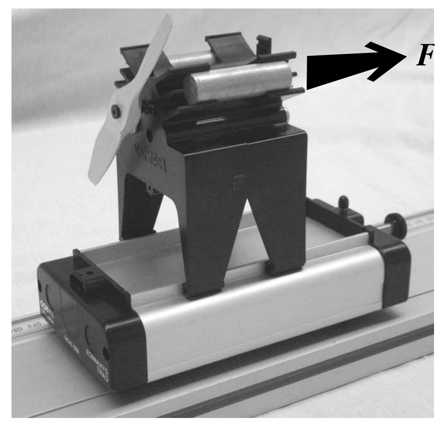


Figure 2.10 A fan cart on a track.

Air resistance and friction

When an object gets going very fast, air resistance, sometimes called "drag", becomes important and at high speeds can be as big as the propelling force. At this point the momentum doesn't increase any more, and the object travels at constant speed. When the speed of an object is low, air resistance is small, so we can make the approximation that the net force is due solely to the fan and is nearly constant.

Friction between the object and the surface on which it moves can also be important. Friction is usually independent of speed, so this is a constant force. The friction between the cart and the track is relatively low, so it is reasonable to say it is approximately zero in this case.

Other interacting objects

If we choose the fan cart as the system, the air is not the only external object that interacts with the system. The Earth exerts a downward gravitational force on the cart, and the track exerts an upward force, as shown in Figure 2.11. We must consider these forces as well, because they contribute to the net external force on the fan cart.

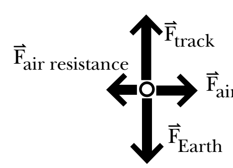


Figure 2.11 Forces on a moving fan cart.

If we see the cart rolling horizontally along the track, it is clear that the y component of its momentum remains zero and does not change:

$$\Delta p_y = 0$$

? Using the Momentum Principle, what can we conclude about the relative magnitudes of the force on the cart by the Earth and the force on the cart by the track?

According to the Momentum Principle the y component of the net force on the cart is zero. Evidently these two forces must be exactly equal in magnitude, and opposite in direction.

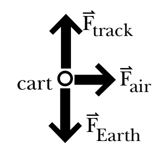
You may wonder how the track "knows" exactly how much upward force to exert on the cart. Basically, the cart compresses the track slightly, and the compressed track pushes up, like a spring. Understanding this in detail requires a microscopic view, discussed in Chapter 4.

Example: A fan cart (1D, several forces, constant net force)

Suppose you have a fan cart whose mass is 400 grams (0.4 kg), and with the fan turned on, the force acting on the cart due to the air and friction with the track, is $\langle 0.2, 0, 0\rangle$ N and constant. You give the cart a shove, and you release the cart at position $\langle 0.5, 0, 0\rangle$ m with initial velocity $\langle 1.2, 0, 0\rangle$ m/s. What is the momentum of the cart 3 seconds later? What is its velocity at this time?

We choose the initial time to be just after you release the cart, so your hand no longer exerts a force on the cart.

System: the cart (including the fan). System is indicated by a circle on the diagram.
Surroundings: the Earth, the track, the air
Initial time: see comment in margin



Momentum Principle

Since the y component of the cart's momentum does not change, we know that the y component of the net force must be zero, and $|\vec{F}_{\text{track}}| = |\vec{F}_{\text{Earth}}|$.

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t = \vec{p}_i + (\vec{F}_{\text{track}} + \vec{F}_{\text{Earth}} + \vec{F}_{\text{air}})(\Delta t)$$
$$\vec{p}_f = \vec{p}_i + \left\langle 0.2, \left(|\vec{F}_{\text{track}}| - |\vec{F}_{\text{Earth}}|\right), 0\right\rangle \text{N}(3\,\text{s})$$
$$\vec{p}_f = \langle 0.4\,\text{kg}1.2\,\text{m/s}, 0, 0\rangle + \langle 0.2, 0, 0\rangle\,\text{N}(3\,\text{s})$$
$$\vec{p}_f = \langle 1.08, 0, 0\rangle\,\text{kg}\cdot\text{m/s}$$
$$\vec{v}_f \approx \frac{\vec{p}_f}{m} = \frac{\langle 1.08, 0, 0\rangle\,\text{kg}\cdot\text{m/s}}{0.4\,\text{kg}} = \langle 2.7, 0, 0\rangle\,\text{m/s}$$

We can use a large time interval Δt because the force isn't changing very much in either magnitude or direction.

Since $v \ll c$, we can use the approximation that $\vec{p} \approx m\vec{v}$.

Check: Speed increased; reasonable since force was in same direction as momentum.

Example: Fast proton (1D, constant net force, relativistic)

A proton in a particle accelerator is moving with velocity $\langle 0.96c, 0, 0\rangle$, so the speed is $0.96 \times 3 \times 10^8$ m/s $= 2.88 \times 10^8$ m/s. A constant electric force is applied to the proton to speed it up, $\vec{F}_{\text{net}} = \langle 5 \times 10^{-12}, 0, 0\rangle$ N. What is the proton's speed as a fraction of the speed of light after 20 nanoseconds ($1\,\text{ns} = 1 \times 10^{-9}\,\text{s}$)?

System: the proton
Surroundings: electric charges in the accelerator

Momentum Principle:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$
$$\langle p_{fx}, 0, 0\rangle = \langle \gamma_i m v_{ix}, 0, 0\rangle + (\langle 5 \times 10^{-12}, 0, 0\rangle\,\text{N})(20 \times 10^{-9}\,\text{s})$$
$$p_{fx} = \frac{1}{\sqrt{1 - \left(\frac{0.96c}{c}\right)^2}}\left(1.7 \times 10^{-27}\,\text{kg}\right)\left(0.96 \times 3 \times 10^8\,\text{m/s}\right) + \left(1 \times 10^{-19}\,\text{N}\cdot\text{s}\right)$$
$$p_{fx} = \left(1.75 \times 10^{-18}\,\text{kg}\cdot\text{m/s}\right) + \left(1 \times 10^{-19}\,\text{N}\cdot\text{s}\right)$$
$$= 1.85 \times 10^{-18}\,\text{kg}\cdot\text{m/s}$$

$$\frac{v_{fx}}{c} = \frac{\dfrac{p_{fx}}{mc}}{\sqrt{1+\left(\dfrac{p_{fx}}{mc}\right)^2}}$$

(See Section 2.13, page 65; obtaining v from p when the speed is near the speed of light.)

$$\text{Evaluate } \frac{p_{fx}}{mc} = \frac{1.85\times10^{-18}\,\text{kg}\cdot\text{m/s}}{\left(1.7\times10^{-27}\,\text{kg}\right)\left(3\times10^{8}\,\text{m/s}\right)} = 3.62 \text{ (no units)}$$

$$\frac{v_{fx}}{c} = \frac{3.62}{\sqrt{1+3.62^2}} = 0.964 \text{ (no units)}$$

Although the magnitude of the momentum increased from 1.75×10^{-18} kg · m/s to 1.85×10^{-18} kg · m/s, the speed didn't increase very much, because the proton's initial speed, $0.96c$, was already close to the cosmic speed limit, c. Because the speed hardly changed, the distance the proton moved during the 20 ns was approximately equal to:
$(0.96\times3\times10^{8}\text{ m/s})(20\times10^{-9}\text{ s}) = 5.8\text{ m}$

2.X.3 A hockey puck is sliding along the ice with nearly constant momentum $\langle 10, 0, 5\rangle$ kg · m/s when it is suddenly struck by a hockey stick with a force $\langle 0, 0, 2000\rangle$ N that lasts for only 3 milliseconds (3×10^{-3}) s. What is the new momentum of the puck?

2.X.4 You were driving a car with velocity $\langle 25, 0, 15\rangle$ m/s. You quickly turned and braked, and your velocity became $\langle 10, 0, 18\rangle$ m/s. The mass of the car was 1000 kg. What was the (vector) change in momentum $\Delta\vec{p}$ during this maneuver? Pay attention to signs. What was the (vector) impulse applied to the car by the ground?

2.X.5 In the previous exercise, if the maneuver took 3 seconds, what was the average net (vector) force $\vec{F}_{\text{net}}$ that the ground exerted on the car?

2.X.6 A truck driver slams on the brakes and the momentum changes from $\langle 9\times10^4, 0, 0\rangle$ kg · m/s to $\langle 5\times10^4, 0, 0\rangle$ kg · m/s in 4 seconds due to a constant force of the road on the wheels of car. As a vector, write the force exerted by the road.

2.X.7 At a certain instant, a particle is moving in the $+x$ direction with momentum $+10$ kg · m/s. During the next 0.1 s, a constant force $\langle -6, 3, 0\rangle$ N acts on the particle. What is the momentum of the particle at the end of this 0.1 s interval?

2.4 UPDATING POSITION IF MOMENTUM IS CHANGING

In the fan cart example (page 48) the momentum, and therefore the velocity, of the fan cart is changing with time. If we want to predict where the fan cart will be at the end of the three second time interval, we need to use an appropriate value for the average velocity of the cart.

We can't calculate the average velocity directly from the definition

$$\vec{v}_{\text{avg}} \equiv \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{\Delta t}$$

because the final location $\vec{r}_f$ of the cart is unknown. Since in this case we do not have any other information that would allow us to determine the average velocity, we need to find an approximate value for the average velocity in this situation. There are two possible approaches:

- If the change in velocity is very small over the time interval of interest, we can use the final velocity (or the initial velocity) as an approximation to the average velocity
- We can try to estimate an average value for the velocity

In the fan cart example on page 48, the final velocity of the cart is $\langle 2.7, 0, 0\rangle$ m/s, which is more than twice the initial velocity of $\langle 1.2, 0, 0\rangle$ m/s. Neither the final nor the initial velocity is a good approximation to the average velocity.

? What might be a better approximation for $\vec{v}_{\text{avg}}$ in this case?

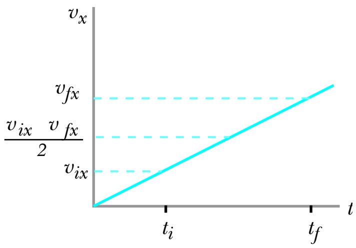


Figure 2.12 v_x is changing at a constant rate (linearly with time), so the arithmetic average is equal to $v_{\text{avg},x}$.

Let's consider using the arithmetic average:

$$v_{\text{avg},x} \approx \frac{(v_{ix} + v_{fx})}{2} \text{ is an approximation for } v_{\text{avg},x} = \frac{\Delta x}{\Delta t}$$

The arithmetic average lies between the two extremes. For example, the arithmetic average of 6 and 8 is $(6+8)/2 = 14/2 = 7$, halfway between 6 and 8.

The arithmetic average is often a good approximation, but it is not necessarily equal to the true average $v_{\text{avg},x} = \Delta x/\Delta t$. The arithmetic average does not give the true average unless v_x is changing at a constant rate, which is the case only if the net force is constant, as it happens to be for a fan cart (see Figure 2.12). The proof that $v_{\text{avg},x} = (v_{ix} + v_{fx})/2$ when v_x changes at a constant rate is given in optional Section 2.11 at the end of this chapter. (The proof is more complicated than one might expect.)

? When is the arithmetic average a poor approximation for $\vec{v}_{\text{avg}}$?

If you drive 50 mi/hr for four hours, and then 20 mi/hr for an hour, you go 220 miles, and your average speed is $(220\,\text{mi})/(5\,\text{hr}) = 44\,\text{mi/hr}$, whereas the arithmetic average is $(50+20)/2 = 35\,\text{mi/hr}$ (see Figure 2.13 for another such example). In situations where the force is not constant, we have to choose short enough time intervals that the velocity is nearly constant during the brief Δt.

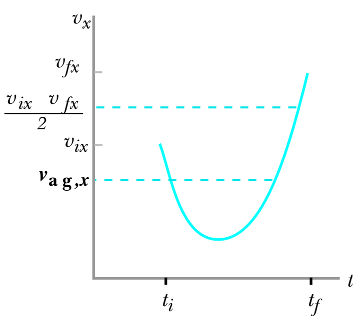


Figure 2.13 v_x is not changing linearly with time. In this case the arithmetic average is much higher than the true average value of $v_{\text{avg},x}$.

APPROXIMATE AVERAGE VELOCITY

$$v_{\text{avg},x} \approx \frac{(v_{ix} + v_{fx})}{2} \text{ is an approximation for } v_{\text{avg},x} = \frac{\Delta x}{\Delta t}$$

Exactly true only if the net force is constant (linear change in v).

May be a poor approximation if net force is not constant.

Now that we have a way to calculate the x component of the average velocity in this special situation (constant x component of force), we can predict the position of the fan cart at the end of the time interval:

$$v_{\text{avg},x} = \frac{(v_{ix} + v_{fx})}{2} = \frac{(1.2+2.7)}{2}\,\text{m/s}$$
$$\vec{v}_{\text{avg}} = \langle 1.95, 0, 0\rangle\,\text{m/s}$$
$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t = \langle 0.5, 0, 0\rangle\,\text{m} + \langle 1.95, 0, 0\rangle \frac{\text{m}}{\text{s}}(3\,\text{s})$$
$$\vec{r}_f = \langle 6.35, 0, 0\rangle\,\text{m}$$

The net force on the cart is constant, so this calculation of average velocity gives the correct value.

2.5 PREDICTING MOTION: CONSTANT FORCE

We can now combine the Momentum Principle with the position update formula to predict the future motion of an object, if the net force on the object is approximately constant over the time interval of interest. The basic approach is:

1. Apply the Momentum Principle (this requires specifying a system and interacting objects) to find the new momentum
2. Use the momentum to find the new position
3. Check reasonableness of results

In ordinary life, there are very few situations in which the net force on a moving system is truly constant, because objects interact with so many other objects at all times. However, it is sometimes useful to simplify a situation by considering the net force to be approximately constant.

Special case: An analytical solution

In the preceding examples we have obtained a numerical solution for the final momentum and position of an object. In a few very special cases it is possible to obtain an "analytical," or algebraic solution. The case of a constant net force is one of these special cases. The analytical solution is an equation that gives the position of an object as a function of time.

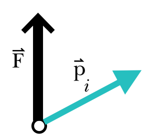
To find an analytical solution we follow the procedure stated above, but we use symbolic expressions instead of numbers. We will solve the problem generally, and then apply our solution to particular situations.

Example: 2D motion, constant net force

An object which is initially in motion is subject to a constant force. Its initial momentum has nonzero x and y components, but the force acts only in the y direction. (For example, this could be a ball thrown slowly enough that air resistance is negligible; it could be a helicopter taking off, or an electron moving through a television tube, or a ball thrown on the airless Moon.) The initial location of the object is $\langle x_i, y_i, 0\rangle$, and its initial velocity is $\langle v_{ix}, v_{iy}, 0\rangle$. The object moves slowly compared to the speed of light. Predict the velocity and position of the object after a time Δt. We'll assume the constant force acts in the $+y$ direction, but later we will be able to use our results for the case where there is a constant force that acts in the $-y$ direction, as is the case with the gravitational force near the Earth's surface.

1. Momentum principle:

System: the object
Surroundings: Constant net force $\langle 0, F_y, 0\rangle$



A possible situation

$$\begin{aligned}
\vec{p}_f &= \vec{p}_i + \vec{F}_{\text{net}}\Delta t \\
\langle p_{fx}, p_{fy}, 0\rangle &= \langle p_{ix}, p_{iy}, 0\rangle + \langle 0, F_y, 0\rangle\Delta t \\
\langle p_{fx}, p_{fy}, 0\rangle &= \langle p_{ix}, (p_{iy} + F_y\Delta t), 0\rangle \\
p_{fx} &= p_{ix} \\
p_{fy} &= (p_{iy} + F_y\Delta t) \\
p_{fz} &= 0
\end{aligned}$$

Since the net force acts only in the y direction, the x and z components of momentum do not change. Only the y component of momentum (or velocity) is changed by this force.

2. Position update ($v \ll c$, so $\vec{p} \approx m\vec{v}$)

$$\begin{aligned}
mv_{fx} &= mv_{ix} \\
v_{fx} &= v_{ix} \\
mv_{fy} &= mv_{iy} + F_y\Delta t \\
v_{fy} &= v_{iy} + \left(\frac{F_y}{m}\right)\Delta t \\
mv_{fz} &= 0 \\
v_{fz} &= 0 \\
v_{\text{avg},x} &= \frac{v_{ix} + v_{fx}}{2} = v_{ix} \\
v_{\text{avg},y} &= \frac{\left(v_{iy} + \left(v_{iy} + \left(\frac{F_y}{m}\right)\Delta t\right)\right)}{2} = v_{iy} + \frac{1}{2}\left(\frac{F_y}{m}\right)\Delta t \\
v_{\text{avg},z} &= 0 \\
\vec{r}_f &= \vec{r}_i + \vec{v}_{\text{avg}}\Delta t
\end{aligned}$$

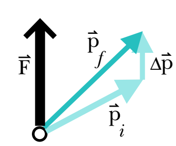


Figure 2.14 Change in momentum of the object during a time interval Δt. The y component of momentum is affected by a force in the y direction.

In this case the arithmetic average gives the correct value, because the velocity is changing at a constant rate.

$$x_f = x_i + v_{ix}\Delta t$$
$$y_f = y_i + v_{iy}\Delta t + \frac{1}{2}\left(\frac{F_y}{m}\right)(\Delta t)^2$$
$$z_f = 0$$

3. Units check: all terms have units of meters.

Component of momentum perpendicular to force

One of the most important things to note about this result is that the x component of momentum was completely unaffected by a force in the y direction. This is a consequence of the vector nature of the Momentum Principle.

Air resistance

There are very few real-world situations in which the net force on a moving object is truly constant. The results of the previous solution are useful only if we can reasonably say that the net force on a moving object is approximately constant.

One significant factor in determining the motion of objects near the Earth is air resistance, sometimes called drag. You have probably felt the effects of air resistance yourself, for example, when coasting downhill on a bicycle. The air resistance force on a moving object depends on the speed of the object, so as the speed of an object changes, the air resistance force changes too, and the net force on the object also changes—it is not constant. Additionally, the direction of the air resistance force changes, since the direction of this force is always opposite to the direction of motion. Note that this is very different from the constant force used in the preceding analysis.

A low-density object such as a styrofoam ball experiences air resistance that is comparable to the small gravitational force on the ball, so air resistance is important unless the styrofoam ball is moving very slowly (air resistance is small at low speeds and big at high speeds, as you may have experienced if you put your hand out the window of a car). At low speeds a baseball, which has a fairly high density, moves with negligible air resistance. But at the speed that a professional pitcher can throw a baseball (about 100 mi/hr or 44 m/s), a baseball goes only about half as far in air as it would in a vacuum, because air resistance is large at this high speed (Figure 2.15).

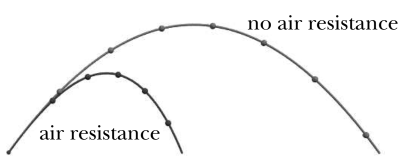


Figure 2.15 The trajectory of a baseball thrown at high speed (around 100 mph, or 44 m/s), ignoring air resistance (top curve) and including the effect of air resistance (bottom curve). The dots indicate the ball's position at equal time intervals. The ball travels about half as far in air as it would in a vacuum.

We will discuss air resistance in more detail in Chapter 6.

Magnitude of the gravitational force near the Earth's surface

The gravitational force on an object of mass m near the surface of the Earth is approximately mg, where g is a positive constant $g = +9.8$ N/kg. We will discuss this further in the next chapter.

Example: A ball with negligible air resistance

A ball of mass 500 g is initially on the ground, at location $\langle 0, 0, 0\rangle$ m, and you kick it with initial velocity $\langle 3, 7, 0\rangle$ m/s. (a) Where will the ball be half a second later? (b) At what time will the ball hit the ground? Make the approximation that air resistance is negligible, and use the previous analytical result for motion with a constant force.

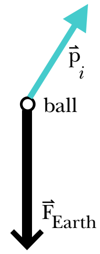
1. Momentum principle

Pick initial and final times so that during the time interval Δt only the Earth exerts a force on the ball.

System: ball
Surroundings: Earth (neglecting air resistance)
Initial time: just after the kick
Final time: just before hitting the ground

2. Position update

(a) Using results from the analytical solution in the previous example:



$$\vec{F}_{net} = \langle 0, -mg, 0\rangle, \text{ so } F_{net,y} = -mg$$
$$x_f \approx x_i + v_{ix}\Delta t$$
$$x_f \approx (0 + (3\,\text{m/s})(0.5\,\text{s}) = 1.5\,\text{m})$$
$$y_f \approx y_i + v_{iy}\Delta t + \frac{1}{2}\left(\frac{(-mg)}{m}\right)(\Delta t)^2$$
$$y_f = 0 + (7\,\text{m/s})(0.5\,\text{s}) - \frac{1}{2}(9.8\,\text{N/kg})(0.5\,\text{s})^2 = 2.275\,\text{m}$$
$$\vec{r}_f = \langle 1.5, 2.275, 0\rangle\,\text{m}$$

In this case, the mass of the ball cancels, because the gravitational force is proportional to mass.

3. Check: correct units. Ball has moved in appropriate direction.

(b) At the instant the ball hits the ground, $y_f = 0$, so

$$0 = 0 + v_{iy}\Delta t + \frac{1}{2}(-g)(\Delta t)^2$$

Solving this quadratic equation for the unknown time Δt, we find two possible values:

$$\Delta t = 0 \text{ and } \Delta t = \frac{2v_{iy}}{g}$$

The first value, $\Delta t = 0$, corresponds to the initial situation, when the ball is near the ground, just after the kick. The second value is the time when the ball returns to the ground, just before hitting:

$$\Delta t = \frac{2(7\,\text{m/s})}{(9.8\,\text{N/kg})} = 1.43\,\text{s}$$

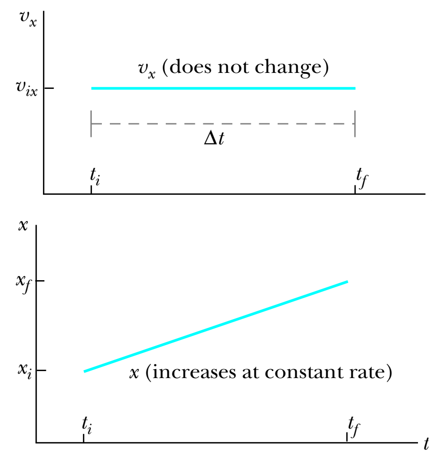


Figure 2.16 Motion graphs for the thrown ball. Top: v_x vs. t, bottom: x vs. t. Note that v_x does not change because the net force acted only in the y direction.

? Could we use these equations for x and y as a function of time to find the location of the ball 10 seconds after you kick it?

No. Our result would be that the ball was far underground, since

$$y_f = (7\,\text{m/s})(10\,\text{s}) - \frac{1}{2}(9.8\,\text{N/kg})(10\,\text{s})^2 = -420\,\text{m}$$

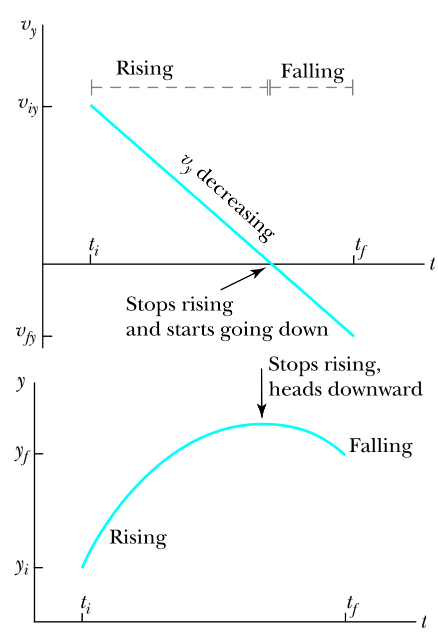
which is not physically reasonable! (The ball would have hit the ground and stopped before 10 seconds had passed, due to other interactions not included in our model.)

Graphs of motion

Figure 2.16 and Figure 2.17 show graphs of position and velocity components vs. time for the ball in the preceding example. In Figure 2.16 the first graph, v_x vs. t, is simply a horizontal line, because v_x doesn't change, since there is no x component of force. The graph of x is a straight line (second graph), rising if v_x is positive. Note that the slope of the x vs. t graph is equal to v_x.

In Figure 2.17 the graph of v_y is a falling straight line (third graph), because the y component of the force is $-mg$, which constantly makes the y component of momentum decrease. At some point the y component of momentum decreases to zero, at the top of the motion, after which the ball heads downward, with negative v_y. The graph of y vs. time t is an inverted parabola (fourth graph), since the equation for y is a quadratic function in the time.

Note that the slope of the y vs. t graph (Figure 2.17) at any time is equal to v_y at that time. In particular, when the slope is zero (at the maximum



This is y vs. time t. It is not the path.

Figure 2.17 Motion graphs for the thrown ball. Top: v_y vs. t; bottom: y vs. t.

y), v_y is momentarily equal to zero. Before that point the slope is positive, corresponding to $v_y > 0$, and after that point the slope is negative, corresponding to $v_y < 0$.

The actual path of the ball, the graph of y vs. x (Figure 2.18), is also an inverted parabola (the bottom graph). Since x increases linearly with t, whether we plot y vs. t or y vs. x we'll see a similar curve. The scale factor along the horizontal axis is different, of course (meters instead of seconds).

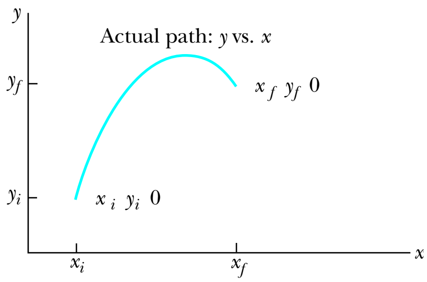


Figure 2.18 The actual trajectory of the thrown ball, with negligible air resistance (y vs. x).

2.X.8 A ball is kicked from a location $\langle 9, 0, -5\rangle$ m (on the ground) with initial velocity $\langle -10, 13, -5\rangle$ m/s.
(a) What is the velocity of the ball 0.6 seconds after being kicked?
(b) What is the location of the ball 0.6 seconds after being kicked?
(c) At what time does the ball hit the ground?
(d) What is the location of the ball when it hits the ground?

2.6 PROBLEMS OF GREATER COMPLEXITY

So far all of the examples we have considered have involved finding a change in momentum (and position), given a known force acting over a known time interval. The following problems require you to find either the duration of an interaction (time interval), or the force exerted during an interaction. These large problems involve several steps in reasoning.

Example: Strike a hockey puck

In Figure 2.19 a 0.4 kg hockey puck is sliding along the ice with velocity $\langle 20, 0, 0\rangle$ m/s. As the puck slides past location $\langle 1, 0, 2\rangle$ m on the rink, a player strikes the puck with a sudden force in the $+z$ direction, and the hockey stick breaks. Some time later, the puck's position on the rink is $\langle 13, 0, 21\rangle$ m. When we pile weights on the side of a hockey stick we find that the stick breaks under a force of about 1000 N (this is roughly 250 pounds; a force of one newton is equivalent to a force of about a quarter of a pound, approximately the weight of a small apple).

(a) Make a sketch of the path of the puck before and after it is hit. (b) For approximately how much time $\Delta t_{\text{contact}}$ was the hockey stick in contact with the puck? Evidently the contact time is quite short, since you hear a short, sharp crack. State what approximations and/or simplifying assumptions you make in your analysis.

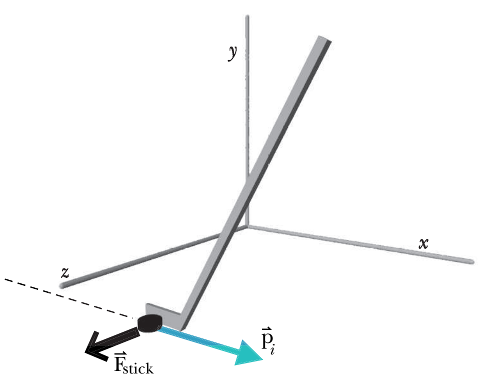
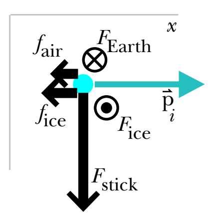


Figure 2.19 A hockey stick hits a puck as it slides by.

1. Momentum principle

System: the hockey puck
Surroundings: Earth, ice, hockey stick, air



Top view, looking down on the ice. ⊗ means "into page" and ⊙ means "out of page". By convention a lower case f usually denotes a frictional force.

Initial time: when stick first makes contact with puck
Final time: when stick breaks

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

$$\langle p_{fx}, 0, p_{fz}\rangle = \langle p_{ix}, 0, 0\rangle + \langle -f_{\text{ice}} - f_{\text{air}}, (F_{\text{ice}} - mg), F_{\text{stick}}\rangle \Delta t_{\text{contact}}$$

x component: $p_{fx} = p_{ix} - (f_{\text{ice}} + f_{\text{air}})\Delta t_{\text{contact}}$

$$f_{\text{ice}} \approx 0 \text{ and } f_{\text{air}} \approx 0, \text{ so}$$

$$p_{fx} \approx p_{ix} + (0)\Delta t_{\text{contact}} \approx p_{ix} \text{ so no change in } p_x$$

y component: $0 = 0 + (F_{\text{ice}} - mg)\Delta t_{\text{contact}}$ therefore $F_{\text{ice}} = mg$

z component: $p_{fz} = F_{\text{stick}}\Delta t_{\text{contact}}$

$$p_{fz} = F_{\text{stick}}\Delta t_{\text{contact}} = (1000\,\text{N})\Delta t_{\text{contact}}$$

Approximations and simplifying assumptions:
Ice exerts little force in the x or z directions (low sliding friction); negligible air resistance.
Force of stick roughly constant during $\Delta t_{\text{contact}}$.
The puck doesn't move very far during the contact time.
After contact, velocity is nearly constant.
There are two unknown quantities in the z component equation: p_{fz} and $\Delta t_{\text{contact}}$. We need another equation to find p_{fz}. Let's try the position update equation.

2. Position update

Initial time: when stick breaks
Final time: when puck is at location $\langle 13, 0, 21\rangle$ m

(a) The path of the puck looks something like Figure 2.20.

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t_{\text{slide}}$$
$$\langle 13, 0, 21\rangle\,\text{m} = (\langle 1, 0, 2\rangle\,\text{m}) + \langle 20\,\text{m/s}, 0, v_z\rangle\Delta t_{\text{slide}}$$

x component: $(13\,\text{m}) = (1\,\text{m}) + (20\,\text{m/s})\Delta t_{\text{slide}}$

$$\Delta t_{\text{slide}} = (12\,\text{m})/(20\,\text{m/s}) = 0.6\,\text{s}$$

Δt_{slide} is the time during which the puck slides after the impact.

y component: $0 = 0 + 0\Delta t_{\text{slide}}$ so $0 = 0$

The y component equation doesn't give any useful information in this case.

z component: Since $\Delta t_{\text{slide}} = 0.6\,\text{s}$:

$$(21\,\text{m}) = (2\,\text{m}) + v_z(0.6\,\text{s})$$
$$v_z = (19\,\text{m})/(0.6\,\text{s}) = 31.7\,\text{m/s}$$

1*. Back to the Momentum Principle

Now we know p_{fz}, and can use it to solve for $\Delta t_{\text{contact}}$.

$$p_{fz} = (1000\,\text{N})\Delta t_{\text{contact}} \text{ where } p_{fz} \approx mv_{fz} \text{ (since } v \ll c)$$
$$(0.4\,\text{kg})(31.7\,\text{m/s}) = (1000\,\text{N})\Delta t_{\text{contact}}$$
$$\text{(b) } \Delta t_{\text{contact}} = (0.4\,\text{kg})(31.7\,\text{m/s})/(1000\,\text{N}) = 0.013\,\text{s}$$

3. Check: Units okay (contact time is in seconds). Time reasonable? The contact time is very short; this is consistent with the sound you hear (a sharp crack).

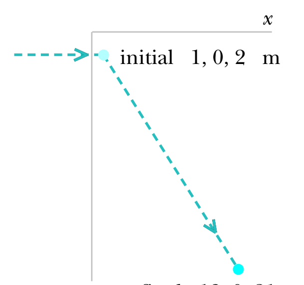


Figure 2.20 The x component of the momentum (and velocity) hardly changes, but the z component of momentum (and velocity) changes quickly from zero to some final value when the puck is hit.

Further discussion

? How good were our assumptions?

The neglect of sliding friction and air resistance is probably pretty good, since a hockey puck slides for long distances on ice with nearly constant speed.

We know the hockey stick exerts a maximum force of $F_{\text{stick}} = 1000\,\text{N}$, because we observe that the stick breaks. We approximate the force as nearly constant during contact. Actually, this force grows quickly from zero at first contact to 1000 N, then abruptly drops to zero when the stick breaks.

The final approximation is somewhat questionable. Although 0.013 s is a short time, the puck moves $(20\,\text{m/s})(0.013\,\text{s}) = 0.26\,\text{m}$ (a bit less than one foot) in the x direction during this time. Also during this time v_z increases from 0 to 31.7 m/s, with an average value of about 15.8 m/s, so the z displacement is about $(15.8\,\text{m/s})(0.013\,\text{s}) = 0.2\,\text{m}$ during contact. On the other hand, these displacements aren't very large compared to the displacement from $\langle 1, 0, 2\rangle$ m to $\langle 13, 0, 21\rangle$ m, so our result isn't terribly inaccurate due to this approximation. Nevertheless, a more accurate sketch of the path of the puck should show a bend as in Figure 2.21.

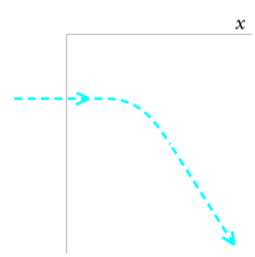


Figure 2.21 A more accurate overhead view of the path of the hockey puck, showing the bend during impact.

Even though our analysis of the stick contact time (0.013 s) isn't exact, it is adequate to get a reasonably good determination of this short time, something that we wouldn't know without using the Momentum Principle and the position update formula. The very short duration of the impact explains why we hear a sharp, short crack.

Estimating times

Real-world problems often require the estimation of one or more quantities. Estimating masses or distances is usually not difficult, but most people find it more challenging to estimate time durations, especially if the times are very short. It is common to guess what seems a rather short time, such as a second, or half a second. However, we found in the previous example that the contact time between a puck and a hockey stick was significantly shorter than this—about a hundredth of a second.

A useful, systematic way to estimate a short time is to use the relationship between velocity and position (a.k.a. the position update formula).

> ? Suppose we had guessed that the contact time between the stick and the puck was about one second. We know that the original speed of the puck was 20 m/s. At this speed, how far would the puck have traveled in 1 s?

$$(20\,\mathrm{m/s})(1\,\mathrm{s}) = 20\,\mathrm{m}$$

It is clear that the puck did not move 20 meters during the impact. We must conclude that the impact took much less than 1 second.

Suppose we guess that the puck may have slid 10 cm along the stick during the impact:

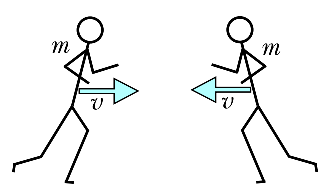
$$\Delta t \approx \frac{(0.10\,\mathrm{m})}{(20\,\mathrm{m/s})} = 0.005\,\mathrm{s}$$

This estimate of the contact time differs from our result by only a factor of around 2, instead of a factor of 100, so it is a much better estimate. Using an object's speed and estimating the distance traveled during an interval allows us to come up with a much better estimate of interaction times than we would otherwise get.

Example: **Colliding students**

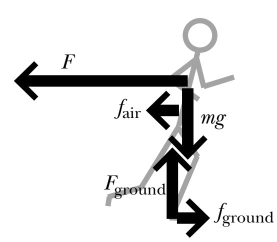
This problem is rather ill-defined and doesn't seem much like a "textbook" problem. No numbers have been given, yet you're asked to estimate the force of the collision. This kind of problem is typical of the kinds of problems engineers and scientists encounter in their professional work.

Two students who are late for tests are running to classes in opposite directions as fast as they can. They turn a corner, run into each other head-on, and crumple into a heap on the ground. Using physics principles, estimate the force that one student exerts on the other during the collision. You will need to estimate some quantities; give reasons for your choices and provide checks showing that your estimates are physically reasonable.



We could have chosen the student on the right as the system; the analysis would be very similar.

1. Momentum Principle



System: the student on the left
Surroundings: Earth, ground, other student, air.
Initial time: just before impact
Final time: when speeds become zero
Simplifying assumptions:

- Students have same mass and same speed
- f_{ground} and f_{air} are negligible compared to F

$$\langle 0,0,0\rangle = \langle p_{ix},0,0\rangle + \langle -F - f_{\text{ground}} - f_{\text{air}}, (F_{\text{ground}} - mg), 0\rangle \Delta t$$

$$x \text{ component: } 0 = p_{ix} - (F + f_{\text{ground}} + f_{\text{air}})\Delta t$$

$$0 \approx p_{ix} - (F)\Delta t$$

y component: $0 = (F_{\text{ground}} - mg)\Delta t$, so $F_{\text{ground}} = mg$
z component: $0 = 0$
Since $0 = p_{ix} - F\Delta t$, $mv = F\Delta t$ (since $v \ll s$)

Estimates: See comments in margin. Mass of the student is around 60 kg. The student's average speed during the impact is 3 m/s. Perhaps the student's body is compressed by about 5 cm during the collision, which will be painful.

2. Position update (during collision)

$$\Delta x = v_{\text{avg},x}\Delta t$$
$$\Delta t = \Delta x / v_{\text{avg},x} = (0.05\,\text{m})/(3\,\text{m/s}) = 0.017\,\text{s}$$

Since $\Delta v_x = (0 - 6)\,\text{m/s}$ and $|\Delta p_x| = |F_x \Delta t|$,

$$|F_x| = \left|\frac{\Delta p_x}{\Delta t}\right| \approx \left|\frac{(60\,\text{kg})(-6\,\text{m/s})}{(0.01\,\text{s})}\right| = 21000\,\text{N}$$

3. Check

- Units check (force is in newtons, collision time is in seconds)
- Is the result reasonable? The contact time is very short, as expected. Is the force reasonable or not? See discussion below.

The student's heels are pushed to the left along the ground due to the collision, and the resisting ground pushes to the right. We'll assume that this force is much smaller than the force exerted by the other student, because the student's shoes can slip.

The y and z component equations don't yield any useful information in this situation.

Notes on estimates

A 1 kg mass weighs 2.2 lb.

An Olympic sprinter can run the 100 meter dash in less than 10 seconds, so 10 m/s is an upper limit. A brisk walking speed is around 2 m/s, so 6 m/s is an intermediate value.

During the collision the student's speed decreases from 6 m/s to 0 m/s, so his or her average speed is about 3 m/s.

Further discussion

21000 N is a very large force. For example, the gravitational force on a 60 kg student (the "weight") is only about $(60\,\text{kg})(9.8\,\text{N/kg}) \approx 600\,\text{N}$. The force of the impact is about 35 times the weight of the student! It's like having a stack of 35 students sitting on you. If the students hit heads instead of stomachs, the squeeze might be less than 1 cm, and the force would be over 5 times as large! This is why heads can break in such a collision.

Our result of 21000 N shows why collisions are so dangerous. Collisions involve very large forces acting for very short times, giving impulses of ordinary magnitude.

Force has units (kg)(m/s)/s, which has the units of momentum (kg · m/s) divided by s, which is correct for a force (change of momentum divided by time).

How good were our approximations?

We made the following approximations and simplifying assumptions:

- We estimated the running speed from known 100 m dash records.
- We estimated the masses of the students.
- We assumed that the horizontal component of the force of the ground on the bottom of the student's shoe was small compared to the force exerted by the other student. Now that we find that the impact force is huge, this assumption seems quite good.
- We made the approximation that the impact force was nearly constant during the impact, so what we've really determined is an average force.
- We assumed that the students had similar masses and similar running speeds, to simplify the analysis. If this is not the case, the analysis is significantly more complicated, but we would still find that the impact force is huge.

You might object that with all these estimates and simplifying assumptions the final result of 21000 N for the impact force is not useful. It is certainly the case that we don't have a very accurate result. But nevertheless we gained valuable information, that the impact force is *very* large. Before doing this analysis based on the Momentum Principle, we had no idea of whether the force was small compared to the student's weight, comparable, or much bigger. Now we have a quantitative result that the force is about 35 times the weight of a student, and we can appreciate why collisions are so dangerous.

2.7 PHYSICAL MODELS

Our model of the colliding student situation is good enough for many purposes. However, we left out some aspects of the actual motion. For example, we mostly ignored the flexible structure of the students, how much their shoes slip on the ground during the collision, etc. We have also quite sensibly neglected the gravitational force of Mars on the students, because it is so tiny compared to the force of one student on the other.

Making and using models is an activity central to physics, and the criteria for a "good" physical model will depend on how we intend to use the model. In fact, one of the most important problems a scientist or engineer faces is deciding what interactions must be included in a model of a real physical, chemical, or biological system, and what interactions can reasonably be ignored.

Order of magnitude estimates

If we neglect some effects, we say that we are constructing a simplified model of the situation. A useful model should omit extraneous detail but retain the most important features of the real-world situation. When we make many estimates, our goal is often to determine the order of magnitude of the answer: is the force on a colliding student closer to 0.0001 newton or 10,000 newtons? Knowing the order of magnitude to expect in an answer can be critically important in solving a real problem, designing an experiment, or designing a crash helmet. The designer of a crash helmet needs an approximate value for the maximum force it must withstand. Actual collisions will vary, so there is no "right" or simple answer.

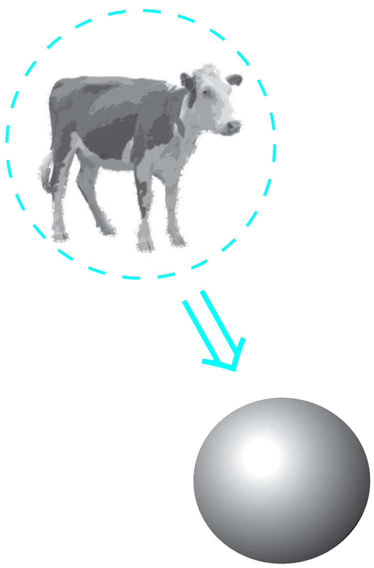


Figure 2.22 An old physics joke begins, "Consider a spherical cow . . . " Sometimes this degree of idealization is actually appropriate.

Idealized models

Some models are "idealized," by which we mean they involve simple, clean, stripped-down situations, free of messy complexities (Figure 2.22). "Ideally," a ball will roll forever on a level floor, but a real ball rolling on a real floor eventually comes to a stop. An "ideal" gas is a fictitious gas in which the molecules don't interact at all with each other, as opposed to a real gas whose molecules do interact, but only when they come close to each other. A model of a single Earth orbiting a Sun is an idealized model because it leaves out the effects of the other bodies in the Solar System.

The behavior of idealized models allows us to investigate simple patterns of motion, and learn what factors are important in determining these patterns. Once we understand these factors, we can revise and extend our models, including more interactions and complexities, to see what effects these have.

An important aspect of physical modeling is that we will engage in making appropriate approximations to simplify the messy, real-world situation enough to permit (approximate) analysis using Newton's laws. Actually, using Newton's laws is itself an example of modeling and making approximations, because we are neglecting the effects of quantum mechanics and of general relativity (Einstein's treatment of gravitation). Newton's laws are only an approximation to the way the world works, though frequently an extremely good one.

2.8 SYSTEMS CONSISTING OF SEVERAL OBJECTS

Reciprocity

When you push on a spring, compressing it, you exert a force on the spring. The compressed spring also exerts a force on your hand. It turns out that the force exerted by your hand on the spring is equal in magnitude (though opposite in direction) to the force exerted by the spring on your hand. This "reciprocity" of forces is a fundamental property of the electric interaction between the electrons and protons in the atoms of your hand and the electrons and protons in the atoms making up the spring. Similarly, the gravitational force that a falling apple exerts on the Earth is just as big as the gravitational force the Earth exerts on the apple. We will say more about the reciprocity of electric and gravitational forces in the next chapter. (Interestingly, reciprocity does not always apply to magnetic forces.)

External forces

In our applications of the Momentum Principle up to this point, we have usually chosen a single object as the system of interest. In some situations, however, it can be very useful to choose a system that consists of two or more interacting objects. This is a legitimate choice of system; we will show that the Momentum Principle applied to a system of two or more objects says that the change in the *total* momentum of the system (Figure 2.23) can be determined by finding the net *external* impulse applied to the system:

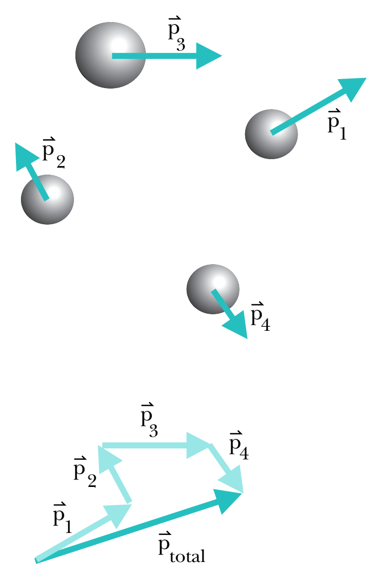


Figure 2.23 The total momentum of the system of four objects is the sum of the individual momenta of each object.

MOMENTUM PRINCIPLE FOR MULTIPARTICLE SYSTEMS

$$\Delta\vec{p}_{\text{total}} = \left(\vec{p}_{\text{total},f} - \vec{p}_{\text{tot al},i}\right) = \vec{F}_{\text{net}}\Delta t$$

Where:

$\vec{p}_{\text{total}} \equiv \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$ sum of momenta of all objects in the system

$\vec{F}_{\text{net}} \equiv \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$ sum of all external forces on the system

This is an interesting statement of the Momentum Principle; it implies that if we know the external forces acting on a multi-object system, we can draw conclusions about the change of momentum of the system over some time interval without worrying about any of the details of the interactions of the objects with each other. This can greatly simplify the analysis of the motion of some very complex systems. It should not be surprising that internal forces alone cannot change a system's momentum. If they could, you could lift yourself off the ground by pulling up on your own feet!

We have implicitly been using the multiparticle version of the Momentum Principle when we have treated macroscopic objects like humans, spacecraft, and planets as if they were single pointlike objects.

Proof of The Momentum Principle for multiparticle systems

In a three-particle system (Figure 2.24) we show all of the forces acting on each particle, where the lower case $\vec{f}$'s are forces the particles exert on each other (so-called "internal" forces), and the upper case $\vec{F}$'s are forces exerted by objects in the surroundings that are not shown and are not part of our chosen system (these are so-called "external" forces, such as the gravitational attraction of the Earth, or a force that you exert by pulling on one of the particles).

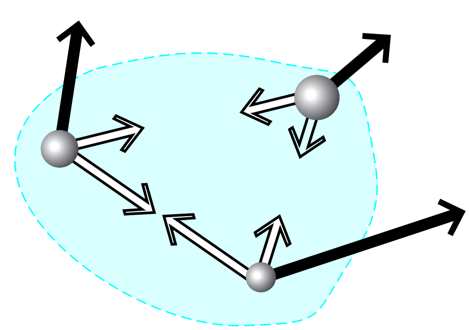


Figure 2.24 External and internal forces acting on a system of three particles.

We will use the following shorthand notation: $\vec{f}_{1,3}$ will denote the force exerted on particle 1 by particle 3; $\vec{F}_{2,\text{surr}}$ will denote the force on particle 2 exerted by objects in the surroundings, and so on. We start by applying the Momentum Principle to each of the three particles separately:

$$\Delta\vec{p}_1 = \left(\vec{F}_{1,\,\mathrm{surr}} + \vec{f}_{1,2} + \vec{f}_{1,3}\right)\Delta t$$
$$\Delta\vec{p}_2 = \left(\vec{F}_{2,\,\mathrm{surr}} + \vec{f}_{2,1} + \vec{f}_{2,3}\right)\Delta t$$
$$\Delta\vec{p}_3 = \left(\vec{F}_{3,\,\mathrm{surr}} + \vec{f}_{3,1} + \vec{f}_{3,2}\right)\Delta t$$

Nothing new so far. But now we add up these three equations. That is, we create a new equation by adding up all the terms on the left sides of the three equations, and adding up all the terms on the right sides, and setting them equal to each other:

$$\begin{aligned}\Delta\vec{p}_1 + \Delta\vec{p}_2 + \Delta\vec{p}_3 = \big(&\vec{F}_{1,\,\mathrm{surr}} + \vec{f}_{1,2} + \vec{f}_{1,3} + \\ &\vec{F}_{2,\,\mathrm{surr}} + \vec{f}_{2,1} + \vec{f}_{2,3} + \\ &\vec{F}_{3,\,\mathrm{surr}} + \vec{f}_{3,1} + \vec{f}_{3,2}\big)\Delta t\end{aligned}$$

Many of these terms cancel. By the principle of reciprocity (see page 59), which is obeyed by gravitational and electric interactions, we have this:

$$\vec{f}_{1,2} = -\vec{f}_{2,1}$$
$$\vec{f}_{1,3} = -\vec{f}_{3,1}$$
$$\vec{f}_{2,3} = -\vec{f}_{3,2}$$

Thanks to reciprocity, all that remains after the cancellations is this:

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 + \Delta\vec{p}_3 = \left(\vec{F}_{1,\,\mathrm{surr}} + \vec{F}_{2,\,\mathrm{surr}} + \vec{F}_{3,\,\mathrm{surr}}\right)\Delta t$$

The total momentum of the system is $\vec{P}_{\mathrm{tot}} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$, so we have:

$$\Delta\vec{P}_{\mathrm{tot}} = \vec{F}_{\mathrm{net,\,surr}}\Delta t$$

The importance of this equation is that reciprocity has eliminated all of the internal forces (the forces that the particles in the system exert on each other); internal forces cannot affect the motion of the system as a whole. All that matters in determining the rate of change of (total) momentum is the net external force. The equation has exactly the same form as the Momentum Principle for a single particle.

In a later chapter we will see that the total momentum can be expressed as $\vec{P}_{\mathrm{tot}} = M_{\mathrm{total}}\vec{v}_{\mathrm{center\ of\ mass}}$, where the center of mass is a mathematical point obtained from a weighted average of the masses in the system.

Moreover, if an object is a sphere whose density is only a function of radius, the object exerts a gravitational force on other objects as though the sphere were a point particle. We can predict the motion of a star or a planet or an asteroid as though it were a single point particle of large mass.

2.9 COLLISIONS: NEGLIGIBLE EXTERNAL FORCES

An event is called a "collision" if it involves an interaction that takes place in a relatively short time and has a large effect on the momenta of the objects compared to the effects of other interactions during that short time. A collision does not necessarily involve actual physical contact between objects, which may be interacting via long distance forces like the gravitational force or the electric force. For example, a spacecraft is deflected as it passes close to Mars on its way to Jupiter, and Mars exerts a large gravitational force for a relatively short time. During that short time the effects of the other planets are negligible.

Often it can be useful to analyze collisions by choosing a system that includes all of the colliding objects, as we will see in the following example.

Example:

Two lumps of clay travel through the air toward each other, at speeds much less than the speed of light (Figure 2.25), rotating as they move. When the lumps collide they stick together. The mass of lump 1 is 0.2 kg and its initial velocity is $\langle 6, 0, 0\rangle$ m/s, and the mass of lump 2 is 0.5 kg and its initial velocity is $\langle -5, 4, 0\rangle$ m/s. What is the final velocity of the stuck-together lumps?

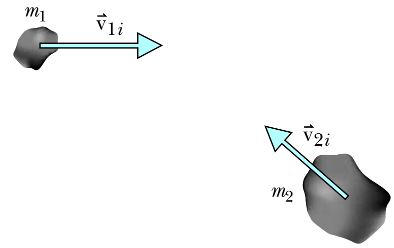


Figure 2.25 Two lumps of clay traveling through the air, just before colliding.

1. Momentum Principle

System: both lumps
Surroundings: air, Earth

$$\begin{aligned}
\vec{p}_{\text{total},f} &= \vec{p}_{\text{total},i} + \vec{F}_{\text{net}}\Delta t \\
\vec{p}_{\text{total},f} &\approx m_1\vec{v}_{1i} + m_2\vec{v}_{2i} + \langle 0,0,0\rangle\Delta t \\
(m_1+m_2)\vec{v}_f &= m_1\vec{v}_{1i} + m_2\vec{v}_{2i} \\
\vec{v}_f &= \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1+m_2} \\
\vec{v}_f &= \frac{(0.2\text{ kg})\langle 6,0,0\rangle + (0.5\text{ kg})\langle -5,4,0\rangle}{(0.7\text{ kg})}\text{ m/s} \\
\vec{v}_f &= \langle -1.86, 2.86, 0\rangle\text{ m/s}
\end{aligned}$$

$v \ll c$; neglect air resistance and gravitational force (which is negligible compared to the contact force during the very short time of the collision).

The rotation of the lumps doesn't affect their momentum, and the Momentum Principle still applies. See Figure 2.26.

2. Not necessary to update position

3. Check: units correct.

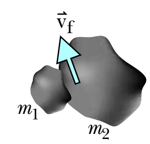


Figure 2.26 The momentum of the stuck-together balls just after the collision is equal to the sum of the initial momenta of the two balls.

Further discussion

By choosing both lumps as the system, we were able to find the final momentum of the system without needing to know anything about the details of the complex forces that the lumps exerted on each other during the collision.

? Why doesn't it matter that the lumps are rotating?

The rotation of an object doesn't affect its momentum; the total momentum of the system is still the sum of the individual momenta of the objects within the system. However, the Momentum Principle does not tell us anything about how fast the stuck-together objects will be rotating; for this, we will have to apply the Angular Momentum Principle, which is discussed in Chapter 10.

The velocities involved are actually the "center-of-mass" velocities of each lump. You can think of this as the velocity of a point at the center of each lump. This concept will be made more quantitative in Chapter 8.

Some problems require more than one principle

Because the balls in the previous example had the same final velocities (Figure 2.26), we had enough information to solve for the final velocity of the stuck-together balls—we had one equation with only one unknown (vector) quantity. In more complex situations, we will find that we sometimes do not have enough information to solve for all of the unknown quantities by applying only the Momentum Principle.

For example, consider a collision between two balls that bounce off each other, as shown in Figure 2.27. The Momentum Principle tells us that

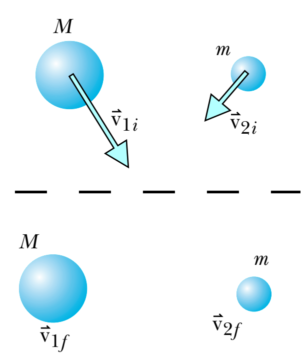


Figure 2.27 The initial momenta of the colliding objects are known, but the final momenta after the collision are unknown.

$$\begin{aligned}
\vec{p}_{\text{total},f} &= \vec{p}_{\text{total},i} \\
\vec{p}_{1f} + \vec{p}_{2f} &= \vec{p}_{1i} + \vec{p}_{2i}
\end{aligned}$$

but we are left with two unknown (vector) quantities, $\vec{p}_{1f}$ and $\vec{p}_{2f}$, and only one equation. We will not be able to analyze problems of this kind fully until we can invoke the Energy Principle (Chapter 5) and the Angular Momentum Principle (Chapter 10), along with the Momentum Principle.

2.X.9 You and a friend each hold a lump of wet clay. Each lump has a mass of 20 grams. You each toss your lump of clay into the air, where the lumps collide and stick together. Just before the impact, the velocity of one lump was $\langle 5, 2, -3\rangle$ m/s, and the velocity of the other lump was $\langle -3, 0, -2\rangle$ m/s. What was the total momentum of the lumps just before impact? What is the momentum of the stuck-together lump just after the impact? What is its velocity?

2.X.10 In outer space, far from other objects, two rocks collide and stick together. Before the collision their momenta were $\langle -10, 20, -5\rangle$ kg · m/s and $\langle 8, -6, 12\rangle$ kg · m/s. What was their total momentum before the collision? What must be the momentum of the combined object after the collision?

2.X.11 At a certain instant, the momentum of a proton is $\langle 3.4 \times 10^{-21}, 0, 0\rangle$ kg · m/s as it approaches another proton which is initially at rest. The two protons repel each other electrically, without coming close enough to touch. When they are once again far apart, one of the protons now has momentum $\langle 2.4 \times 10^{-21}, 1.6 \times 10^{-21}, 0\rangle$ kg · m/s. At this instant, what is the momentum of the other proton?

2.10 CONSERVATION OF MOMENTUM

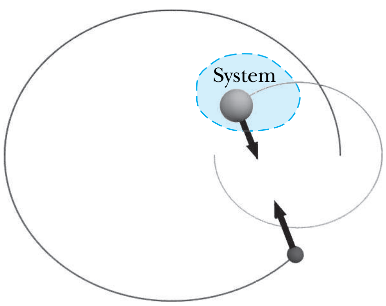


Figure 2.28 A binary star. The gray lines show the trajectories of the individual stars. Choose just one of the stars as the system. The momentum of the system changes due to the external force.

The choice of system affects the detailed form of the Momentum Principle. Consider the case of two stars orbiting each other, a "binary star." If you choose as the system of interest just one of the stars (Figure 2.28), the other star is in the "surroundings" and exerts an external force which changes the first star's momentum.

If on the other hand you choose both stars as your system (Figure 2.29), the surroundings consist of other stars, which may be so far away as to have negligible effects on the binary star. In that case the net external force acting on the system is nearly zero, which makes the analysis simpler.

? What happens to the total momentum of the isolated binary star as time goes by?

The Momentum Principle $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$ reduces in this case to $\vec{p}_f = \vec{p}_i$, which predicts that the total momentum of the system $\vec{p} = \vec{p}_1 + \vec{p}_2$ remains constant in magnitude and direction. This is an important special case: the total momentum of an isolated system, a system with negligible interactions with the surroundings, doesn't change but stays constant.

In a later chapter we will see that the total momentum can be expressed as $M_{\text{total}}\vec{v}_{\text{center of mass}}$, where the center of mass is a mathematical point between the two stars, closer to the more massive star. The velocity of the center of mass does not change but is constant as the binary star drifts through space (or is zero if the binary star's total momentum is zero).

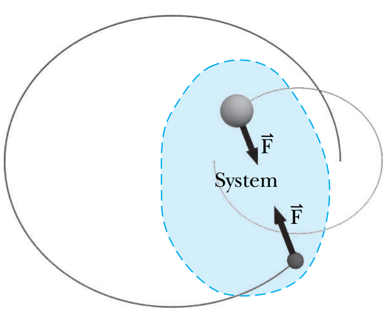


Figure 2.29 A binary star: choose both stars as the system. The momentum of the combined system doesn't change.

One way to think about this result is to say that momentum gained by one star is lost by the other, because the gravitational forces (and impulses) are equal in magnitude but opposite in direction (see the short discussion of reciprocity on page 59). The effect is that the total momentum doesn't change. Let $\vec{F}$ be the force exerted on star 1 by star 2, so $-\vec{F}$ is the force exerted on star 2 by star 1. After a short time interval Δt the new total momentum is this:

$$(\vec{p}_1 + \vec{F}\Delta t) + (\vec{p}_2 - \vec{F}\Delta t) = \vec{p}_1 + \vec{p}_2$$

Momentum lost by one star is gained by the other star. This is a simple example of an important restatement of the Momentum Principle, "conservation of momentum," which says that the change of momentum in a system plus the change of momentum in the surroundings adds up to zero.

CONSERVATION OF MOMENTUM

$$\Delta\vec{p}_{\text{system}} + \Delta\vec{p}_{\text{surroundings}} = \vec{0}$$

In the case of the two stars there are no objects in the surroundings, and no external forces, so the momentum of the system doesn't change. Zero net external impulse ($\vec{F}_{\text{net}}\Delta t$), zero net momentum change. One of the stars can gain momentum (due to a force acting on it), but only if the other star loses the same amount.

Relativistic momentum conservation

Particle accelerators produce beams of particles such as electrons, protons, and pions at speeds very close to the speed of light. When these high-speed particles interact with other particles, experiments show that the total momentum is conserved, but only if the momentum of each particle is defined in the way Einstein proposed, $\vec{p} \equiv \gamma m\vec{v}$. When the speed v approaches the speed of light c, the low-speed approximation $\gamma \approx 1$ ($\vec{p} \approx m\vec{v}$) is not valid (that is, the quantity $m\vec{v}$ is not conserved when the speed v approaches the speed of light c).

What is not conserved?

In later chapters we will see that energy and angular momentum are also conserved quantities. However, most quantities are not conserved quantities. For example, velocity is not conserved in an interaction, as we saw clearly in the collision between the two lumps of clay (Figure 2.25 and Figure 2.26). Temperature is another example of a quantity that is not conserved.

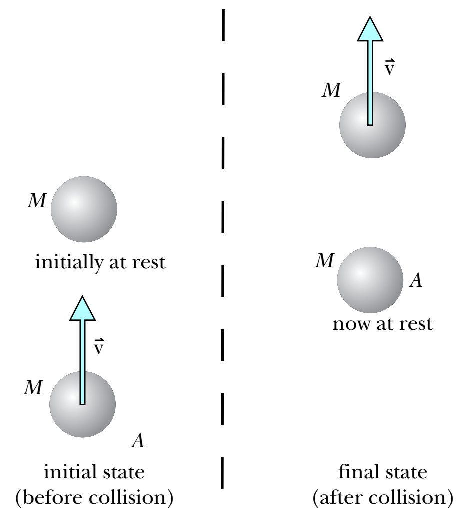


Figure 2.30 A head-on collision between two identical bowling balls, each of mass M.

2.X.12 Consider the head-on collision of two identical bowling balls (Figure 2.30).

(a) Choose a system consisting only of ball A. What is the momentum change of the system during the collision? What is the momentum change of the surroundings?

(b) Choose a system consisting only of ball B. What is the momentum change of the system during the collision? What is the momentum change of the surroundings?

(c) Choose a system consisting of both balls. What is the momentum change of the system during the collision? What is the momentum change of the surroundings?

2.X.13 You hang from a tree branch, then let go and fall toward the Earth. As you fall, the y component of your momentum, which was originally zero, becomes large and negative.

(a) Choose yourself as the system. There must be an object in the surroundings whose y momentum must become equally large, and positive. What object is this?

(b) Choose yourself and the Earth as the system. The y component of your momentum is changing. Does the total momentum of the system change? Why or why not?

2.11 *DERIVATION: SPECIAL CASE AVERAGE VELOCITY

Here are two proofs, one geometric and one algebraic (using calculus), for the following special-case result concerning average velocity:

$$v_{\text{avg},x} = \frac{(v_{ix} + v_{fx})}{2} \text{ \textit{only} if } v_x \text{ changes at a constant rate}$$

Similar results for $v_{\text{avg},y}$ and $v_{\text{avg},z}$

Geometric proof

If $F_{\text{net},x}$ is constant, $p_{fx} = p_{ix} + F_{\text{net},x}\Delta t$ implies that p_x changes at a constant rate. At speeds small compared to the speed of light, $v_x \approx p_x/m$, so a graph of v_x vs. time is a straight line, as in Figure 2.31. Using this graph, we form narrow vertical slices, each of height v_x and narrow width Δt.

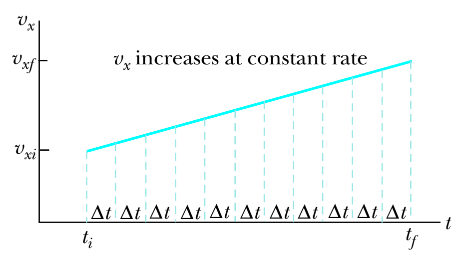


Figure 2.31 Graph of v_x vs. t (constant force), divided into narrow vertical slices each of height v_x and width Δt.

Within each narrow slice v_x changes very little, so the change in position during the brief time Δt is approximately $\Delta x = v_x\Delta t$. Therefore the change in x is approximately equal to the area of the slice of height v_x and width Δt (Figure 2.32).

Figure 2.32 One narrow slice has an area given approximately by $v_x\Delta t$. This is equal to Δx, the displacement of the object.

If we add up the areas of all these slices, we get approximately the area under the line in Figure 2.31, and this is also equal to the total displacement $\Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots = x_f - x_i$. If we go to the limit of an infinite number of slices, each with infinitesimal width, the sum of slices really *is* the area, and this area we have shown to be equal to the change in position. This kind of sum of an infinite number of infinitesimal pieces is called an "integral" in calculus.

The area under the line is a trapezoid, and from geometry we know that the area of a trapezoid is the average of the two bases times the altitude:

$$\textit{area} = \frac{(\textit{top} + \textit{bottom})}{2}(\textit{altitude})$$

Turn Figure 2.31 on its side, as in Figure 2.32, and you see that the top and bottom have lengths v_{ix} and v_{fx}, while the altitude of the trapezoid is the total time $(t_f - t_i)$. Therefore we have the following result:

$$\text{Trapezoid area } = x_f - x_i = \frac{(v_{ix} + v_{fx})}{2}(t_f - t_i)$$

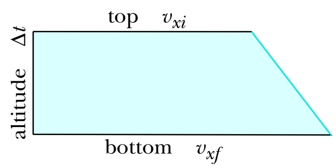


Figure 2.33 The area of the whole trapezoid is equal to the total displacement. The figure in Figure 2.31 has been rotated 90 degrees clockwise.

Dividing by $(t_f - t_i)$, we have this:

$$\frac{x_f - x_i}{t_f - t_i} = \frac{(v_{ix} + v_{fx})}{2}$$

But by definition the x component of average velocity is the change in x divided by the total time, so we have proved that

$$v_{\text{avg},x} = \frac{(v_{ix} + v_{fx})}{2} \text{ \textit{only} if } v_x \text{ changes linearly with time}$$

The proof depended critically on the straight-line ("linear") change in velocity, which occurs if $F_{\text{net},x}$ is constant (and $v \ll c$). Otherwise we wouldn't have a trapezoidal area. That's why the result isn't true in general; it's only true in this important but special case.

Algebraic proof using calculus

An algebraic proof using calculus can also be given. We will use the x component of the derivative version of the Momentum Principle (more about this in a later chapter):

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t \text{ implies that } \frac{\Delta\vec{p}}{\Delta t} = \vec{F}_{\text{net}}$$

In the limit we have

$$\lim_{\Delta t\to 0}\frac{\Delta\vec{p}}{\Delta t} = \frac{d\vec{p}}{dt} = \vec{F}_{\text{net}} \text{ and } \frac{dp_x}{dt} = F_{\text{net},x}$$

If $F_{\text{net},x}$ is a constant, the time derivative of p_x is a constant, so we have

$$p_x = F_{\text{net},x}t + p_{xi} \text{ since } p_x = p_{ix} \text{ when } t = 0$$

You can check this by taking the derivative with respect to time t, which gives the original equation $dp_x/dt = F_{\text{net},x}$. At speeds small compared to the speed of light, $v_x \approx p_x/m$, so we can write

$$v_x = \frac{F_{\text{net},x}}{m}t + v_{ix} \text{ since } v_x = v_{ix} \text{ when } t = 0$$

But the x component of velocity is the rate at which x is changing:

$$v_x = \frac{dx}{dt} = \frac{F_{\text{net},x}}{m}t + v_{ix}$$

Now the question is, can you think of a function of x that has this time derivative? Since the time derivative of t^2 is $2t$, the following formula for x has the appropriate derivative:

$$x = \frac{1}{2}\frac{F_{\text{net},x}}{m}t^2 + v_{ix}t + x_i \text{ since } x = x_i \text{ when } t = 0$$

You can check this by taking the derivative with respect to t, which gives the equation for v_x, since $d(\frac{1}{2}t^2)/dt = t$ and $d(t)/dt = 1$.

The average velocity which we seek is the change in position divided by the total time:

$$v_{\text{avg},x} = \frac{x - x_i}{t} = \frac{1}{2}\frac{F_{\text{net},x}}{m}t + v_{ix} = \frac{1}{2}\left(v_{fx} - v_{ix}\right) + v_{xi}$$

where we have used the equation we previously derived for the velocity:

$$v_{fx} = v_x = \frac{F_{\text{net},x}}{m}t + v_{ix}$$

Simplifying the expression for $v_{\text{avg},x}$ we have the proof:

$$v_{\text{avg},x} = \frac{\left(v_{ix} + v_{fx}\right)}{2} \textit{ only} \text{ if } v_x \text{ changes at a constant rate.}$$

2.12 *Inertial frames

Newton's first law is valid only in an "inertial frame" of reference, one in uniform motion (or at rest) with respect to the pervasive "cosmic microwave background" (see optional discussion at the end of Chapter 1). Since the Momentum Principle is a quantitative version of Newton's first law, we expect the Momentum Principle to be valid in an inertial reference frame, but not in a reference frame that is not in uniform motion. Let's check that this is true.

If you view some objects from a space ship that is moving uniformly with velocity $\vec{v}_s$ with respect to the cosmic microwave background, all of the velocities of those objects have the constant $\vec{v}_s$ subtracted from them, as far as you are concerned. For example, a rock moving at the same velocity as your spacecraft would have $\vec{v}_{\text{rock}} = (\vec{v} - \vec{v}_s) = \vec{0}$ in your reference frame: it would appear to be stationary as it coasted along beside your spacecraft.

With a constant spaceship velocity, we have $\Delta\vec{v}_s = \vec{0}$, and the change of momentum of the moving object reduces to the following (for speeds small compared to c):

$$\Delta\left[m\left(\vec{v} - \vec{v}_s\right)\right] = \Delta(m\vec{v}) - \Delta\left(m\vec{v}_s\right) = \Delta(m\vec{v}) \text{ since } \Delta\left(m\vec{v}_s\right) = \vec{0}$$

$$\text{Therefore, } \Delta\left[m\left(\vec{v} - \vec{v}_s\right)\right] = \Delta(m\vec{v}) = \vec{F}_{\text{net}}\Delta t$$

If the velocity $\vec{v}_s$ of the space ship doesn't change (it represents an inertial frame of reference), the form (and validity) of the Momentum Principle is unaffected by the motion of the space ship.

However, if your space ship increases its speed, or changes direction, $\Delta\vec{v}_s \neq \vec{0}$, an object's motion relative to you changes without any force acting on it. In that case the Momentum Principle is not valid for the object, because you are not in an inertial frame. Although the Earth is not an inertial frame because it rotates, and goes around the Sun, it is close enough to being an inertial frame for many everyday purposes.

2.13 *Velocity and Momentum at High Speeds

If $v \ll c$, $\vec{p} \approx m\vec{v}$ and $\vec{v} \approx \vec{p}/m$. But at high speed it is more complicated to determine the velocity from the (relativistic) momentum. Here is a way to solve for $\vec{v}$ in terms of $\vec{p}$:

$$|\vec{p}| = \frac{1}{\sqrt{1 - \left(|\vec{v}|/c\right)^2}}m|\vec{v}|$$

Divide by m and square: $\dfrac{|\vec{p}|^2}{m^2} = \dfrac{|\vec{v}|^2}{1 - \left(|\vec{v}|/c\right)^2}$

Multiply by $\left(1 - \left(|\vec{v}|/c\right)^2\right)$: $\dfrac{|\vec{p}|^2}{m^2} - \left(\dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = |\vec{v}|^2$

Collect terms: $\left(1 + \dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = \dfrac{|\vec{p}|^2}{m^2}$

$$|\vec{v}| = \frac{|\vec{p}|/m}{\sqrt{1 + \left(\frac{|\vec{p}|}{mc}\right)^2}}$$

But since $\vec{p}$ and $\vec{v}$ are in the same direction, we can write this:

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1 + \left(\frac{|\vec{p}|}{mc}\right)^2}}$$

2.14 *Measurements and units

Using the Momentum Principle requires a consistent way to measure length, time, mass, and force, and a consistent set of units. We state the definitions of the standard Système Internationale (SI) units, and we briefly discuss some subtle issues underlying this choice of units.

Units: meters, seconds, kilograms, coulombs, and newtons

Originally the meter was defined as the distance between two scratches on a platinum bar in a vault in Paris, and a second was 1/86,400th of a "mean solar day." Now however the second is defined in terms of the frequency of light emitted by a cesium atom, and the meter is defined as the distance light travels in 1/299,792,458th of a second, or about 3.3×10^{-9} seconds (3.3 nanoseconds). The speed of light is defined to be exactly 299,792,458 m/s (very close to 3×10^8 m/s). As a result of these modern redefinitions, it is really speed (of light) and time that are the internationally agreed-upon basic units, not length and time.

By international agreement, one kilogram is the mass of a platinum block kept in that same vault in Paris. As a practical matter, other masses are compared to this standard kilogram by using a balance-beam or spring weighing scale (more about this in a moment). The newton, the unit of force, is defined as that force which acting for 1 second imparts to 1 kilogram a velocity change of 1 m/s. We could make a scale for force by calibrating the amount of stretch of a spring in terms of newtons.

The coulomb, the SI unit of electric charge, is defined in terms of electric currents. The charge of a proton is 1.6×10^{-19} coulomb.

Some subtle issues

What we have just said about SI units is sufficient for practical purposes to predict the motion of objects, but here are some questions that might bother you. Is it legitimate to measure the mass that appears in the Momentum Principle by seeing how that mass is affected by gravity on a balance-beam scale? Is it legitimate to use the Momentum Principle to define the units of force, when the concept of force is itself associated with the same law? Is this all circular reasoning, and the Momentum Principle merely a definition with little content? Here is a chain of reasoning that addresses these issues.

Measuring inertial mass

When we use balance-beam or spring weighing scales to measure mass, what we're really measuring is the "gravitational mass," that is, the mass that appears in the law of gravitation and is a measure of how much this object is affected by the gravity of the Earth. In principle, it could be that this "gravitational mass" would be different from the so-called "inertial mass"—the mass that appears in the definition of momentum. It is possible to compare the inertial masses of two objects, and we find experimentally that inertial and gravitational mass seem to be entirely equivalent.

Here is a way to compare two inertial masses directly, without involving gravity. Starting from rest, pull on the first object with a spring stretched by some amount s for an amount of time Δt, and measure the increase of speed Δv_1. Then, starting from rest, pull on the second object with the same spring stretched by the same amount s for the same amount of time Δt, and measure the increase of speed Δv_2. We *define* the ratio of the inertial masses as $m_1/m_2 = \Delta v_2/\Delta v_1$. Since one of these masses could be the standard kilogram kept in Paris, we now have a way of measuring inertial mass in kilograms. Having defined inertial mass this way, we find experimentally that the Momentum Principle is obeyed by both of these objects in all situations, not just in the one special experiment we used to compare the two masses.

Moreover, we find to extremely high precision that the inertial mass in kilograms measured by this comparison experiment is exactly the same as the gravitational mass in kilograms obtained by comparing with a standard kilogram on a balance-beam scale (or using a calibrated spring scale), and that it doesn't matter what the objects are made of (wood, copper, glass, etc.). This justifies the convenient use of ordinary weighing scales to determine inertial mass.

Is this circular reasoning?

The definitions of force and mass may sound like circular reasoning, and the Momentum Principle may sound like just a kind of definition, with no real content, but there is real power in the Momentum Principle. Forget for a moment the definition of force in newtons and mass in kilograms. The experimental fact remains that any object if subjected to a single force by a spring with constant stretch experiences a change of momentum (and velocity) proportional to the duration of the interaction. Note that it is not a change of *position* proportional to the time (that would be a constant speed), but a change of *velocity*. That's real content. Moreover, we find that the change of velocity is proportional to the amount of stretch of the spring. That too is real content.

Then we find that a different object undergoes a different rate of change of velocity with the same spring stretch, but after we've made one single comparison experiment to determine the mass relative to the standard kilogram, the Momentum Principle works in all situations. That's real content.

Finally come the details of setting standards for measuring force in newtons and mass in kilograms, and we use the Momentum Principle in helping set these standards. But logically this comes after having established the law itself.

2.15 SUMMARY

The Momentum Principle

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t \text{ (for a short enough time interval } \Delta t)$$

$$\text{Update form: } \vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

Momentum Principle for multiparticle systems

$$\Delta\vec{p}_{\text{total}} = \vec{F}_{\text{net,ext}}\Delta t$$

Where $\vec{p}_{\text{total}} \equiv \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots = M_{\text{total}}\vec{v}_{\text{center of mass}}$ and $\vec{F}_{\text{net,ext}} \equiv \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$, the sum of all external forces acting on the system

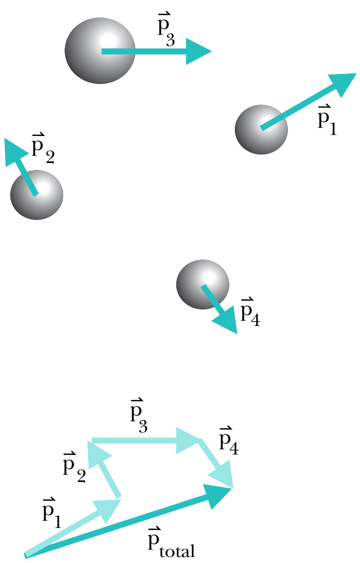


Figure 2.34 The total momentum of the system of four objects is the sum of the individual momenta of each object.

Conservation of momentum

$$\Delta\vec{p}_{\text{system}} + \Delta\vec{p}_{\text{surroundings}} = \vec{0}$$

To analyze the motion of a real-world system, several steps are required:

1. Momentum Principle

Choose a system, consisting of a portion of the Universe.

Make a list of objects in the surroundings that exert significant forces on the chosen system, and make a labeled diagram showing the external forces exerted by the objects in the surroundings.

Apply the Momentum Principle to the chosen system:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

For each term in the Momentum Principle, substitute any values you know.

2. Apply the position update formula, if necessary:

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t$$

3. Check for reasonableness (units, direction, magnitude, etc.).

System is a portion of the Universe acted on by the *surroundings*.

Force is a quantitative measure of interactions; units are newtons.

Impulse is the product of force times time $\vec{F}\Delta t$; momentum change equals net impulse (the impulse due to the net force).

Physical models are tractable approximations/idealizations of the real world.

Special case result for average velocity:

$v_{\text{avg},x} = \dfrac{(v_{ix} + v_{fx})}{2}$ *only* if v_x changes linearly with time

Similar results for $v_{\text{avg},y}$ and $v_{\text{avg},z}$

2.16 REVIEW QUESTIONS

The Momentum Principle

2.RQ.14 An object is moving in the $+x$ direction. Which, if any, of the following statements do you know must be false?

A. The net force on the object is in the $+x$ direction.
B. The net force on the object is in the $-x$ direction.
C. The net force on the object is zero.

2.RQ.15 You observe three carts moving to the left:

Cart A moves at nearly constant speed.
Cart B moves to the left, gradually speeding up.
Cart C moves to the left, gradually slowing down.

Which cart or carts, if any, experience a net force to the left?

2.RQ.16 The x component of the momentum of an object is observed to increase with time:

At $t = 0\,\text{s}$, $p_x = 30\,\text{kg}\cdot\text{m/s}$
At $t = 1\,\text{s}$, $p_x = 40\,\text{kg}\cdot\text{m/s}$
At $t = 2\,\text{s}$, $p_x = 50\,\text{kg}\cdot\text{m/s}$
At $t = 3\,\text{s}$, $p_x = 60\,\text{kg}\cdot\text{m/s}$

What can you conclude about the x component of the net force acting on the object during this time?

A. $F_{\text{net},x} = 0$
B. $F_{\text{net},x}$ is constant.
C. $F_{\text{net},x}$ is increasing with time.
D. Not enough information is given.

2.RQ.17 At a certain instant, a particle is moving in the $+x$ direction with momentum $+10\,\text{kg}\cdot\text{m/s}$. During the next $0.1\,\text{s}$, a constant force acts on the particle: $F_x = -6\,\text{N}$, and $F_y = +3\,\text{N}$. What is the magnitude of the momentum of the particle at the end of this $0.1\,\text{s}$ interval?

2.RQ.18 At $t = 12.0$ seconds an object with mass 2 kg was observed to have a velocity of $\langle 10, 35, -8\rangle$ m/s. At $t = 12.3$ seconds its velocity was $\langle 20, 30, 4\rangle$ m/s. What was the average (vector) net force acting on the object?

2.RQ.19 A proton has mass 1.7×10^{-27} kg. What is the magnitude of the impulse required to increase its speed from $0.990c$ to $0.994c$?

2.RQ.20 In order to pull a sled across a level field at constant velocity you have to exert a constant force. Doesn't this violate Newton's first and second laws of motion, which imply that no force is required to maintain a constant velocity? Explain this seeming contradiction.

Conservation of momentum

2.RQ.21 A bullet traveling horizontally at a very high speed embeds itself in a wooden block that is sitting at rest on a very slippery sheet of ice. You want to find the speed of the block just after the bullet embeds itself in the block.

(a) What should you choose as the system to analyze?
1. The bullet
2. The block
3. The bullet and the block
(b) Which of the following statements is true?
1. After the collision, the speed of the block with the bullet stuck in it is the same as the speed of the bullet before the collision.
2. The momentum of the block with the bullet stuck in it is the same as the momentum of the bullet before the collision.
3. The momentum of the block with the bullet stuck in it is less than the momentum of the bullet before the collision.

2.RQ.22 One kind of radioactivity is called "alpha decay." For example, the nucleus of a radium-220 atom can spontaneously split into a radon-216 nucleus plus an alpha particle (a helium nucleus, containing two protons and two neutrons).

Consider a radium-220 nucleus which is initially at rest. It spontaneously decays, and the alpha particle travels off in the $+z$ direction. What can you conclude about the motion of the new radon-216 nucleus? Explain your reasoning.

A. It is also moving in the $+z$ direction
B. It remains at rest.
C. It is moving in the $-z$ direction.

2.RQ.23 A bowling ball is initially at rest. A ping pong ball moving in the $+z$ direction hits the bowling ball and bounces off it, traveling back in the $-z$ direction.

Consider a time interval Dt extending from slightly before to slightly after the collision.

(a) In this time interval, what is the sign of Δp_z for the system consisting of both balls?
1. Positive
2. Negative
3. Zero—no change in p_z.

(b) In this time interval, what is the sign of Δp_z for the system consisting of the bowling ball alone?
1. Positive
2. Negative
3. Zero—no change in p_z.

2.17 PROBLEMS

2.P.24 In the space shuttle

A space shuttle is in a circular orbit near the Earth. An astronaut floats in the middle of the shuttle, not touching the walls. On a diagram, draw and label

(a) the momentum $\vec{p}_1$ of the astronaut at this instant;
(b) all of the forces (if any) acting on the astronaut at this instant;
(c) the momentum $\vec{p}_2$ of the astronaut a short time Δt later;
(d) the momentum change (if any) $\Delta\vec{p}$ in this time interval.
(e) Why does the astronaut seem to "float" in the shuttle?

It is ironic that we say the astronaut is "weightless" despite the fact that the only force acting on the astronaut is the astronaut's weight (that is, the gravitational force of the Earth on the astronaut).

2.P.25 Crash test

In a crash test, a truck with mass 2200 kg traveling at 25 m/s (about 55 miles per hour) smashes head-on into a concrete wall without rebounding. The front end crumples so much that the truck is 0.8 m shorter than before. What is the approximate magnitude of the force exerted on the truck by the wall? Explain your analysis carefully, and justify your estimates on physical grounds.

2.P.26 Ping-pong ball

A ping-pong ball is acted upon by the Earth, air resistance, and a strong wind. Here are the positions of the ball at several times.
Early time interval:
At $t = 12.35$ s, the position was $\langle 3.17, 2.54, -9.38\rangle$ m
At $t = 12.37$ s, the position was $\langle 3.25, 2.50, -9.40\rangle$ m
Late time interval:
At $t = 14.35$ s, the position was $\langle 11.25, -1.50, -11.40\rangle$ m
At $t = 14.37$ s, the position was $\langle 11.27, -1.86, -11.42\rangle$ m

(a) In the early time interval, from $t = 12.35$ s to $t = 12.37$ s, what was the average momentum of the ball? The mass of the ping-pong ball is 2.7 grams (2.7×10^{-3} kg). Express your result as a vector.

(b) In the late time interval, from $t = 14.35$ s to $t = 14.37$ s, what was the average momentum of the ball? Express your result as a vector.

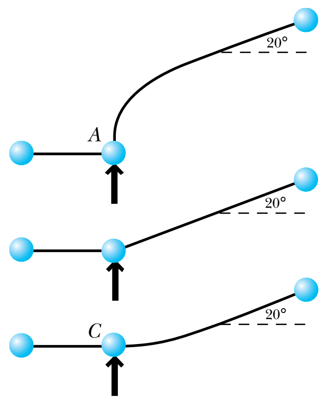
(c) In the time interval from $t = 12.35$ s (the start of the early time interval) to $t = 14.35$ s (the start of the late time interval), what was the average net force acting on the ball? Express your result as a vector.

2.P.27 Proton and HCl molecule

A proton interacts electrically with a neutral HCl molecule located at the origin. At a certain time t, the proton's position is $\langle 1.6 \times 10^{-9}, 0, 0\rangle$ m and the proton's velocity is $\langle 3200, 800, 0\rangle$ m/s. The force exerted on the proton by the HCl molecule is $\langle -1.12 \times 10^{-11}, 0, 0\rangle$ N. At a time $t + (2 \times 10^{-14}\text{ s})$, what is the approximate velocity of the proton?

2.P.28 Kick a soccer ball

A 0.4 kg soccer ball is rolling by you at 3.5 m/s. As it goes by, you give it a kick perpendicular to its path. Your foot is in contact with the ball for 0.002 s. The ball eventually rolls at a 20° angle from its original direction. The overhead view in the diagram below is approximately to scale. The arrow represents the force your toe applies briefly to the soccer ball.



(a) In the diagram, which letter corresponds to the correct overhead view of the ball's path?

(b) Determine the magnitude of the average force you applied to the ball.

2.P.29 Projectile motion

A small dense ball with mass 1.5 kg is thrown with initial velocity $\langle 5, 8, 0\rangle$ m/s at time $t = 0$ at a location we choose to call the origin ($\langle 0, 0, 0\rangle$ m). Air resistance is negligible.

(a) When the ball reaches its maximum height, what is its velocity (a vector)? It may help to make a simple diagram.

(b) When the ball reaches its maximum height, what is t? You know how v_y depends on t, and you know the initial and final velocities.

(c) Between the launch at $t = 0$ and the time when the ball reaches its maximum height, what is the average velocity (a vector)? You know how to determine average velocity when velocity changes at a constant rate.

(d) When the ball reaches its maximum height, what is its location (a vector)? You know how average velocity and displacement are related.

(e) At a later time the ball's height y has returned zero, which means that the average value of v_y from $t = 0$ to this time is zero. At this instant, what is the time t?

(f) At the time calculated in part (e), when the ball's height y returns to zero, what is x? (This is called the "range" of the trajectory.)

(g) At the time calculated in part (e), when the ball's height y returns to zero, what is v_y?

(h) What was the angle to the x-axis of the initial velocity?

(i) What was the angle to the x-axis of the velocity at the time calculated in part (e), when the ball's height y returned to zero?

2.P.30 A free throw in basketball

Determine two different possible ways for a player to make a free throw in basketball. In both cases give the initial speed, initial angle, and initial height of the basketball. The rim of the basket is 10 feet (3.0 m) above the floor. It is 14 feet (4.3 m) along the floor from the free-throw line to a point directly below the center of the basket.

2.P.31 A basketball pass

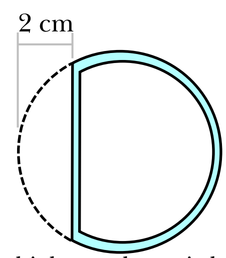
You have probably seen a basketball player throw the ball to a teammate at the other end of the court, 30 m away. Estimate a reasonable initial angle for such a throw, and then determine the corresponding initial speed. For your chosen angle, how long does it take for the basketball to go the length of the court? What is the highest point along the trajectory, relative to the thrower's hand?

2.P.32 The case of the falling flower pot

You are a detective investigating why someone was hit on the head by a falling flowerpot. One piece of evidence is a home video taken in a 4th-floor apartment, which happens to show the flowerpot falling past a tall window. Inspection of individual frames of the video shows that in a span of 6 frames the flowerpot falls a distance that corresponds to 0.85 of the window height seen in the video (note: standard video runs at a rate of 30 frames per second). You visit the apartment and measure the window to be 2.2 m high. What can you conclude? Under what assumptions? Give as much detail as you can.

2.P.33 Tennis ball hits wall

A tennis ball has a mass of 0.057 kg. A professional tennis player hits the ball hard enough to give it a speed of 50 m/s (about 120 miles per hour). The ball hits a wall and bounces back with almost the same speed (50 m/s). As indicated in the diagram, high-speed photography shows that the ball is crushed 2 cm (0.02 m) at the instant when its speed is momentarily zero, before rebounding.



A high-speed tennis ball deforms when it hits a wall.

Making the very rough approximation that the large force that the wall exerts on the ball is approximately constant during contact, determine the approximate magnitude of this force. Hint: Think about the approximate amount of time it takes for the ball to come momentarily to rest. (For comparison note that the gravitational force on the ball is quite small, only about $(0.057\text{ kg})(9.8\text{ N/kg}) \approx 0.6\text{ N}$. A force of 5 N is approximately the same as a force of one pound.)

2.P.34 Mars probe

A small space probe, of mass 240 kg, is launched from a spacecraft near Mars. It travels toward the surface of Mars, where it will land. At a time 20.7 seconds after it is launched, the probe is at the location $\langle 4.30 \times 10^3, 8.70 \times 10^2, 0\rangle$ m, and at this same time its momentum is $\langle 4.40 \times 10^4, -7.60 \times 10^3, 0\rangle$ kg · m/s. At this instant, the net force on the probe due to the gravitational pull of Mars plus the air resistance acting on the probe is $\langle -7 \times 10^3, -9.2 \times 10^2, 0\rangle$ N.

(a) Assuming that the net force on the probe is approximately constant over this time interval, what is the momentum of the probe 20.9 seconds after it is launched?

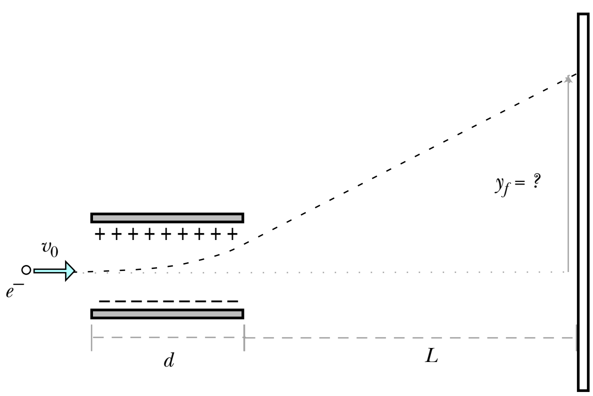
(b) What is the location of the probe 20.9 seconds after launch?

2.P.35 Spacecraft navigation

Suppose you are navigating a spacecraft far from other objects. The mass of the spacecraft is 1.5×10^5 kg (about 150 tons). The rocket engines are shut off, and you're coasting along with a constant velocity of $\langle 0, 20, 0\rangle$ km/s. As you pass the location $\langle 12, 15, 0\rangle$ km you briefly fire side thruster rockets, so that your spacecraft experiences a net force of $\langle 6 \times 10^4, 0, 0\rangle$ N for 3.4 s. The ejected gases have a mass that is small compared to the mass of the spacecraft. You then continue coasting with the rocket engines turned off. Where are you an hour later? Also, what approximations or simplifying assumptions did you have to make in your analysis? Think about the choice of system: what are the surroundings that exert external forces on your system?

2.P.36 Electron motion in a CRT

In a cathode ray tube (CRT) used in oscilloscopes and television sets, a beam of electrons is steered to different places on a phosphor screen, which glows at locations hit by electrons. The CRT is evacuated, so there are few gas molecules present for the electrons to run into. Electric forces are used to accelerate electrons of mass m to a speed $v_0 \ll c$, after which they pass between positively and negatively charged metal plates which deflect the electron in the vertical direction (upward in the diagram, or downward if the sign of the charges on the plates is reversed).



A cathode ray tube.

While an electron is between the plates, it experiences a uniform vertical force F, but when the electron is outside the plates there is negligible force on it. The gravitational force on the electron is negligibly small compared to the electric force in this situation. The length of the metal plates is d, and the phosphor screen is a distance L from the metal plates. Where does the electron hit the screen? (That is, what is y_f?)

2.P.37 The SLAC two-mile accelerator

SLAC, the Stanford Linear Accelerator Center, located at Stanford University in Palo Alto, California, accelerates electrons through a vacuum tube two miles long (it can be seen from an overpass of the Junipero Serra freeway that goes right over the accelerator). Electrons which are initially at rest are subjected to a continuous force of 2×10^{-12} newton along the entire length of two miles (one mile is 1.6 kilometers) and reach speeds very near the speed of light.

(a) Determine how much time is required to increase the electrons' speed from $0.93c$ to $0.99c$. (That is, the quantity $|\vec{v}|/c$ increases from 0.93 to 0.99.)

(b) Approximately how far does the electron go in this time? What is approximate about your result?

2.P.38 Outer space collision

In outer space a small rock with mass 5 kg traveling with velocity $\langle 0, 1800, 0\rangle$ m/s strikes a stationary large rock head-on and bounces straight back with velocity $\langle 0, -1500, 0\rangle$ m/s. After the collision, what is the vector momentum of the large rock?

2.P.39 Two rocks collide

Two rocks collide in outer space. Before the collision, one rock had mass 9 kg and velocity $\langle 4100, -2600, 2800\rangle$ m/s. The other rock had mass 6 kg and velocity $\langle -450, 1800, 3500\rangle$ m/s. A 2 kg chunk of the first rock breaks off and sticks to the second rock. After the collision the rock whose mass is 7 kg has velocity $\langle 1300, 200, 1800\rangle$ m/s. After the collision, what is the velocity of the other rock, whose mass is 8 kg?

2.P.40 Two rocks collide

Two rocks collide with each other in outer space, far from all other objects. Rock 1 with mass 5 kg has velocity $\langle 30, 45, -20\rangle$ m/s before the collision and $\langle -10, 50, -5\rangle$ m/s after the collision. Rock 2 with mass 8 kg has velocity $\langle -9, 5, 4\rangle$ m/s before the collision. Calculate the final velocity of rock 2.

2.P.41 Two rocks stick together

In outer space two rocks collide and stick together. Here are the masses and initial velocities of the two rocks:
Rock 1: mass = 15 kg, initial velocity = $\langle 10, -30, 0\rangle$ m/s
Rock 2: mass = 32 kg, initial velocity = $\langle 15, 12, 0\rangle$ m/s
What is the velocity of the stuck-together rocks after colliding?

2.P.42 Moving the Earth

Suppose all the people of the Earth go to the North Pole and, on a signal, all jump straight up. Estimate the recoil speed of the Earth. The mass of the Earth is 6×10^{24} kg, and there are about 6 billion people (6×10^9).

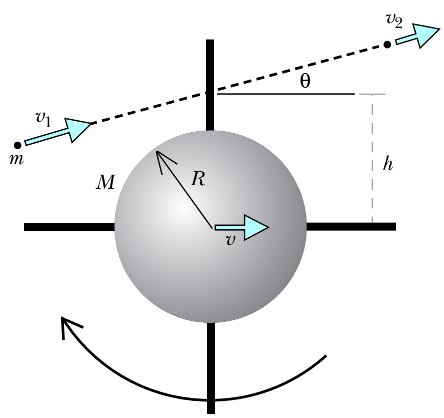
2.P.43 Bullet embeds in block

A bullet of mass m traveling horizontally at a very high speed v embeds itself in a block of mass M that is sitting at rest on a nearly frictionless surface. What is the speed of the block after the bullet embeds itself in the block?

2.P.44 Meteor hits a spinning satellite

A satellite which is spinning clockwise has four low-mass solar panels sticking out as shown. A tiny meteor traveling at high speed rips through one of the solar panels and continues in the same direction but at reduced speed. Afterwards, calculate the

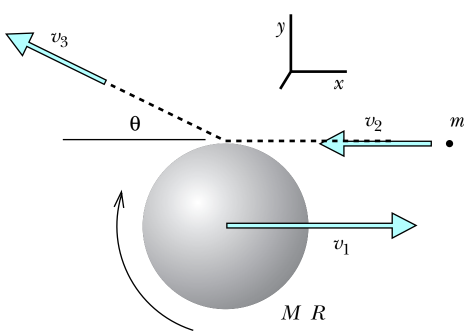
v_x and v_y components of the center-of-mass velocity of the satellite. Initial data are provided on the diagram.



2.P.45 Space junk

A tiny piece of space junk of mass m strikes a glancing blow to a spinning satellite. Before the collision the satellite was moving and rotating as shown in the diagram. After the collision the space junk is traveling in a new direction and moving more slowly. The space junk had negligible rotation both before and after the collision. The velocities of the space junk before and after the collision are shown in the diagram. The satellite has mass M and radius R.

Just after the collision, what are the components of the center-of-mass velocity of the satellite (v_x and v_y)?



2.P.46 Two balls collide

A ball of mass 0.05 kg moves with a velocity $\langle 17, 0, 0\rangle$ m/s. It strikes a ball of mass 0.1 kg which is initially at rest. After the collision, the heavier ball moves with a velocity of $\langle 3, 3, 0\rangle$ m/s.

(a) What is the velocity of the lighter ball after impact?

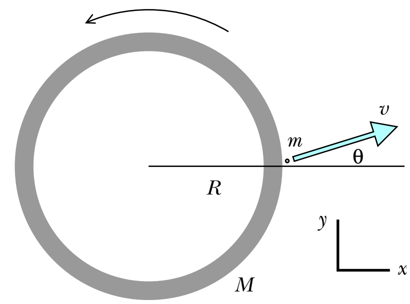
(b) What is the impulse delivered to the 0.05 kg ball by the heavier ball?

(c) If the time of contact between the balls is 0.03 s, what is the force exerted by the heavier ball on the lighter ball?

2.P.47 Space station

A space station has the form of a hoop of radius R, with mass M. Initially its center of mass is not moving, but it is spinning. Then a small package of mass m is thrown by a spring-loaded gun toward a nearby spacecraft as shown; the package has a speed v after launch.

Calculate the center-of-mass velocity of the space station (v_x and v_y) after the launch.



2.P.48 Falling ball

Apply the general results obtained in the full analysis of motion under the influence of a constant force on page 51 to answer the following questions. You hold a small metal ball of mass m a height h above the floor. You let go, and the ball falls to the floor. Choose the origin of the coordinate system to be on the floor where the ball hits, with y up as usual. Just after release, what are y_i and v_{iy}? Just before hitting the floor, what is y_f? How much time Δt does it take for the ball to fall? What is v_{fy} just before hitting the floor? Express all results in terms of m, g, and h. How would your results change if the ball had twice the mass?

2.18 ANSWERS TO EXERCISES

2.X.1 (page 45) $\langle -60, -24, 96\rangle$ N · s, $\langle -60, -24, 96\rangle$ N · s

2.X.2 (page 45) $\langle 0, -2, 0\rangle$ N · s, $\langle 0, -0.667, 0\rangle$ N · s

2.X.3 (page 49) $\vec{p}_f = \langle 10, 0, 11\rangle$ kg · m/s

2.X.4 (page 49) $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \langle -15000, 0, 3000\rangle$ kg · m/s = impulse by ground

2.X.5 (page 49) $\vec{F}_{\text{net}} = \langle -5000, 0, 1000\rangle$ N

2.X.6 (page 49) $\vec{F}_{\text{net}} = \langle -1\times 10^4, 0, 0\rangle$ N

2.X.7 (page 49) $\vec{p}_f = \langle 9.4, 0.3, 0\rangle$ kg · m/s

2.X.8 (page 54) (a) $\langle -10, 7.12, -5\rangle$ m/s

(b) $\langle 3, 6.036, -8\rangle$ m

(c) 2.66 s

(d) $\langle -17.5, 0, -18.3\rangle$ m

2.X.9 (page 62) $\langle 0.04, 0.04, -0.1\rangle$ kg · m/s,
$\langle 0.04, 0.04, -0.1\rangle$ kg · m/s
$\langle 1, 1, -2.5\rangle$ m/s

2.X.10 (page 62) $\langle 18, 14, 7\rangle$ kg · m/s, $\langle 18, 14, 7\rangle$ kg · m/s

2.X.11 (page 62) $\langle 1\times 10^{-21}, -1.6\times 10^{-21}, 0\rangle$ kg · m/s

2.X.12 (page 63) (a) $-m\vec{v}$, $+m\vec{v}$; (b) $+m\vec{v}$, $-m\vec{v}$; (c) $\vec{0}, \vec{0}$

2.X.13 (page 63) (a) The Earth (b) No. The changes in your momentum and the Earths momentum are equal and opposite, and add up to zero. As you move down, the Earth moves up. The Earth gets as much magnitude of momentum as you do, but very little velocity because its mass is so huge ($v \approx p/m$).